Controllability of a hyperbolic and a parabolic system in one dimensional periodic domain

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Declaration

I hereby declare that this thesis is my own work and, to the best of my knowledge, it contains no materials previously published or written by any other person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Kolkata or any other educational institution, except where due acknowledgment is made in the thesis.

May, 2022
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Certificate

This is to certify that the work entitled "Controllability of a linear hyperbolic and a linear parabolic equation in one dimensional periodic domain" embodies the research work done by Manish Kumar under my guidance and supervision for the partial fulfillment of the degree of Master of Sciences, Indian Institute of Science Education and Research, Kolkata, India. It has not previously formed the basis for the award of any Degree, Diploma, Associateship or Fellowship to him.

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Abstract

This thesis mainly studies the controllability aspects of a linear first order hyperbolic equation (transport equation), and a fourth order linear parabolic equation, called Kuramoto-Sivashinsky-Korteweg-de Vries (KS-KdV), in one dimensional periodic spatial domain $(0, 2\pi)$. The controllability of both the systems has been studied by means of localized interior control as well as periodic boundary control acting at the zeroth oder derivatives of the variable.

For the case of transport equation, the exact controllability result at time $T > T_0$, for $T_0 = \frac{2\pi}{|a|}$ has been proved, where a is the velocity, which eventually imply its null controllability. Moreover, for $T < T_0$ it has been shown that the transport equation is not null controllable and hence not exact controllable. While in case of KS-KdV equation the null controllability at any time $T > 0$ has been proved. For the transport equation, the duality approach, more precisely, Carleman estimates has been adopted, while in case of KS-KdV equation, a well known direct method, called the method of moments has been utilized. This method requires an appropriate biorthogonal family, whose existence can be ensured from an existing result.

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Contents

CHAPTER 0

Prerequisites

In this chapter, we will introduce different notions of controllability of differential equations and give the standard existing results in abstract settings. At first in Section 0.1, we will understand the concept of controllability of ordinary differential equations (ODEs), which represents the finite dimensional system and give the relevant available results. In the next Section 0.2, we will brief the theory of semigroups so that we can extend the notion of controllability of finite dimensional systems to infinite dimensional systems. Consequently, we will extend the notion of controllability for abstract partial differential equations(PDEs), which represent infinite dimensional systems in Section 0.3. Finally, we will do a bit of comparison between the controllability results for finite and infinite dimensional systems in the last Section 0.4. This chapter has been borrowed from [1], [2], [3], [4], and so one can follow these references for the more detailed study.

0.1 Finite Dimensional Control Systems (ODEs)

Consider the system of ordinary diiferential equations:

$$\begin{cases} \dfrac{dy}{dt} = Ay(t) + Bq(t), \quad t \in (0, \infty) \\ y(0) = y_0 \end{cases} \tag{0.1.1}$$

where, $y(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^m$ for $t > 0$ represent the state and control , respectively ($\mathbb{R}^n$ is called state space and $\mathbb{R}^m$ is called control space), the initial data $y_0 \in \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$, and $B : \mathbb{R}^m \to \mathbb{R}^n$ are linear maps, i.e., $A \in M_n(\mathbb{R})$ and $B \in M_{n,m}(\mathbb{R})$. Then for given $y_0 \in \mathbb{R}^n, q \in L^2(0, T; R^m)$ the system (0.1.1) has a unique solution $y \in H^1(0, T; \mathbb{R}^n)$ given by

$$y(t) = e^{tA}y_0 + \int_0^t e^{A(t-s)}Bq(s)\, ds, \quad t > 0 \tag{0.1.2}$$

Consider the system (0.1.1) to be posed in $(0, T)$, for some $T > 0$. Then, the adjoint system of the system (0.1.1) can be given as:

$$\begin{cases} \dfrac{d\varphi}{dt} = A\varphi(t) + Bq(t), & t \in (0, T) \\ \varphi(T) = \varphi_T \end{cases} \tag{0.1.3}$$

For understanding the theory, we will consider the following two examples in this chapter.

Example 0.1.1. *Let $T > 0$. Take*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So, here $n = 2$ and $m = 1$. More precisely, the system reads as:

$$\begin{cases} y_1'(t) = y_1(t) + q(t), & t \in (0, T), \\ y_2'(t) = y_2(t), & t \in (0, T), \end{cases}$$

equivalently,

$$\begin{cases} y_1'(t) = y_1(t) + q(t), & t \in (0, T), \\ y_2(t) = y_{20}e^t, & t \in (0, T), \end{cases}$$

where $y_2(0) = y_{20}$, for some $y_{20} \in \mathbb{R}$.

Example 0.1.2. *Let $T > 0$. Consider the controlled differential equation $y'' + y = q$. It can be easily reduced to a coupled system of ODEs as:*

$$\begin{cases} y_1'(t) = y_2(t), & t \in (0, T), \\ y_2'(t) = -y_1(t) + q(t), & t \in (0, T), \end{cases}$$

where $y_1 = y$ So, we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, here also $n = 2$ and $m = 1$.

Now, let us define different notions of controllability:

Definition 0.1.3 (Exact Controllability)**.** *System (0.1.1) is said to be exactly controllable in some given time $T > 0$, if for any $y_0, y_1 \in \mathbb{R}^n$ there exists $q \in L^2(0, T; \mathbb{R}^m)$ such that the solution given by (0.1.2) satisfies $y(T) = y_1$.*

Definition 0.1.4 (Approximate Controllability)**.** *System (0.1.1) is said to be approximately controllable in some given time $T > 0$, if for any $y_0, y_1 \in \mathbb{R}^n$ and for any $\epsilon > 0$ there exists $q_\epsilon \in L^2(0, T; \mathbb{R}^m)$ such that the solution given by (0.1.2) satisfies $|y(T) - y_1| < \epsilon$.*

Definition 0.1.5 (Null Controllability)**.** *System (0.1.1) is said to be null controllable in some given time $T > 0$, if for any $y_0 \in \mathbb{R}^n$, there exists $q \in L^2(0, T; \mathbb{R}^m)$ such that the solution given by (0.1.2) satisfies $y(T) = 0$.*

Remark 0.1.6. *For practical significances, we need $m \leq n$, i.e., we want less than n number of controls to be employed in a control system with n number of states.*

Remark 0.1.7. *Note the following obvious observations:*

- *Clearly, exact controllability of system (0.1.1) implies the null controllability and approximate controllability of the system (0.1.1).*
- *Using the fact that the only dense subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself, one can easily conclude that approximate controllability of system (0.1.1) gives its exact controllability and hence null controllability.*

As we now know that all the above-mentioned notions of controllability for a system of ODEs (0.1.1) are equivalent, so from now we will simply use the phrase "System (0.1.1) (or (A, B)) is controllable in time $T > 0$ " to denote the controllability of (0.1.1) in time $T > 0$. The time $T > 0$ is called the controllability time for the system.

Observation 0.1.8. *Let us now check the controllability of the system considered in Example 0.1.1 and Example 0.1.2.*

1. *In the Example 0.1.1, it is quite clear that the second component of the system cannot be controlled at all, it will have its own dynamics.*
2. *For the case of Example 0.1.2, let (y_{10}, y_{20}) and (y_{11}, y_{21}) be arbitrary points in $\mathbb{R}^2$. Our aim is to find $q \in L^2(0, T; \mathbb{R}^2)$ such that the solution of system satisfies $(y_1(0), y_2(0)) = (y(0), y'(0)) = (y_{10}, y_{20})$ and $(y_1(T), y_2(T)) = (y(T), y'(T)) = (y_{11}, y_{21})$. Let x be a cubic polynomial in t satisfying the conditions $(x(0), x'(0)) = (y_{10}, y_{20})$ and $(x(T), x'(T)) = (y_{11}, y_{21})$. Then the solution y of the ODE with initial condition $(y(0), y'(0)) = (y_{10}, y_{20})$ with the control q as $q(t) = x''(t) + x(t)$, coincides with x and so satisfies $(y(T), y'(T)) = (y_{11}, y_{21})$. Hence, the system is (exact) controllable.*

Moreover, we have an equivalent algebraic criteria to check the controllability of system of ODEs, which is precisely stated in the theorem below.

Theorem 0.1.9 (Kalman rank condition). *(A, B) is controllable (in time $T > 0$) iff $\operatorname{rank}[A|B] = n$, where n is dimension of the state space and $[A|B] \in M_{n,mn}(\mathbb{R})$ given by $[A|B] = [B\ AB\ A^2B\ \cdots\ A^{n-1}B]$.*

Remark 0.1.10. *From the above theorem, it is quite clear that controllability time has no meaning in case of system of ODEs, i.e., if (A, B) is controllable for some time $T > 0$, then it is controllable for any time $T > 0$ as the above algebraic condition does not involve time of controllability. In case of infinite dimensional systems, this need not be true and controllability time is very important, for example, see the controllability results for the transport equation in the next Chapter 2.*

Observation 0.1.11. *Let us now verify the controllability of above examples with the above theorem:*

1. *In case of Example 0.1.1, $[A|B] = [B\ AB] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, which has rank 1 and hence not controllable by the Kalman rank condition, which agrees with Observation 0.1.8.*
2. *In case of Example 0.1.2 $[A|B] = [BAB] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, whose rank is 2, and hence the system is controllable, which agrees with Observation 0.1.8.*

Using the properties of the solution adjoint system (0.1.3), one can also conclude about the controllability of (A, B). Such approach is called duality approach of studying the controllability. More precisely, we have the following theorem:

Theorem 0.1.12. *(Duality method) The following are equivalent:*

(i) The system (0.1.1) (or (A, B)) is controllable.

(ii) For any time $T > 0$, there exist $C_T > 0$ such that the solution φ of (0.1.3) satisfies

$$\int_0^T |B^*\varphi|^2 \, dt \geq C_T |\varphi_T|^2, \forall \varphi_T \in \mathbb{R}^n. \quad \text{[Observability inequality]}$$

(iii) For any time $T > 0$, there exist $C_T > 0$ such that the solution φ of (0.1.3) with φ_T varying over $\mathbb{R}^n$, satisfies

$$\int_0^T |B^*\varphi|^2 \, dt \geq C_T |\varphi(0)|^2. \quad \text{[Observability inequality]}$$

(iv) Fopr any $\varphi_T \in \mathbb{R}^n$, the solution of the adjoint system (0.1.3) satisfies

$$B^*\varphi(t) = 0, \, \forall t \in [0, T] \implies \varphi_T = 0. \quad \text{[Unique continuation principle]}$$

0.2 Semigroup Theory

Let V_1, V_2 denote any Banach spaces. Let A be a linear operator on V_1 with values in V_2, i.e., domain of A is a subspace of V_1 which we denote by $D(A)$ and the range of A, denoted by $R(A)$ is subspace of V_2.

Definition 0.2.1. *Let us recall some definitions from Functional analysis.*

1. *A linear operator $A : D(A) \subset V_1 \to V_2$ is said to be bounded if there exists some $C > 0$ such that*
$$||Aw||_{V_2} \leq C||u||_{V_1}, \quad \forall w \in D(A).$$

2. *If A does not satisfy such inequality, then A is said to be unbounded.*

3. *A is said to be densely defined if $\overline{D(A)} = V_1$.*

4. *A is said to be closed if the graph $G(A) := \{(w, Aw) : w \in D(A)\}$ is closed subspace of $V_1 \times V_2$.*

Motivation: In case of infinite dimensional systems, the matrix A in (0.1.1) is replaced by a linear operator defined on some Banach space as we will see in the next Section 0.3, and the solution will have the same form as in case of finite dimensional case, which is (0.1.2). But we know that e^{tA} is sensible when A is a bounded operator but does not make sense for unbounded operator A. In such situation, we replace e^{tA} by the 'semigroup of operator A', denoted by $S(t)$. This concept of semigroup of operators is generalization of exponential of operators as for bounded operator A, $S(t) = e^{tA}$. All these facts will be made precise in the remaining of the section.

Let V be a Banach space and let $A : V \to V$ be a bounded linear operator. Consider the differential equation:

$$\begin{cases} \frac{dy}{dt}(t) & = Ay(t), \quad t > 0, \\ y(0) & = y_0 \in V. \end{cases} \tag{0.2.1}$$

This system has a unique solution given by $y(t) = e^{tA}y_0$ with the following properties:

- For fixed t, $y_0 \to y(t)$ is a linear map on V.
- $||y(t)|| \leq e^{t||A||}||y_0||$.
- $y(t) \to y_0$ as $t \downarrow 0$ and also $y(0) = y_0$.
- By uniqueness of solution to (0.2.1), if we start with initial data $y(t_0)$ then the solution after time t_1 will be $y(t_1 + t_0)$, which is the solution of (0.2.1) at time $t = t_1 + t_0$.

Definition 0.2.2 (C_0-Semigroup). *Assume V is a Banach space. We say a family of bounded operators $\{S(t)\}_{t\geq 0}$ on V is C_0-Semigroup if it satisfies following:*

(a) $S(0) = I$, where I is the identity operator on V.

(b) $S(t+s) = S(t)S(s), \ \forall\, t, s \geq 0$. [Semigroup property]

(c) For every $y \in V$, $S(t)y \to y$ as $t \downarrow 0$. [continuity w.r.t t]

Let us now state some properties of the C_0-Semigroup $\{S(t)\}_{t\geq 0}$:

- $||S(t)|| \leq Mw^{\omega t}, \ \forall t \geq 0$.
- The mapping from $[0, \infty]$ into V given by $t \to S(t)y$ for any $y \in V$ is a continuous map.

Definition 0.2.3 (Contraction semigroup). *If $||S(t)|| \leq 1$ for all $t \geq 0$ (i.e., $M = 1$ and $\omega = 0$ in the above stated property), then we call $\{S(t)\}_{t\geq 0}$ as contraction C_0-semigroup.*

Definition 0.2.4 (Infinitesimal generator). *The infinitesimal generator of a C_0-semigroup on V, $\{S(t)\}_{t\geq 0}$ is a linear operator A defined on V as*

$$Aw = \lim_{t\downarrow 0} \frac{S(t)w - w}{t}, \ w \in D(A) \text{ with } D(A) = \left\{ w \in V : \lim_{t\downarrow 0} \frac{S(t)w - w}{t} \text{ exists} \right\}.$$

Theorem 0.2.5. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-Semigroup and let A be its infinitesimal generator. Then for $w \in D(A)$, we have:*

$$S(t)w \in C^1\left([0, \infty); V\right) \cap C\left([0, \infty); D(A)\right),$$
$$\text{and} \quad \frac{d}{dt}\left(S(t)w\right) = AS(t)w = S(t)Aw.$$

Remark 0.2.6. *From the above theorem, it is clear that if A is infinitesimal generator of some C_0-semigroup $\{S(t)\}_{t\geq 0}$, then $y(t) = S(t)y_0$ solves the system (0.2.1) if the initial data $y_0 \in D(A)$. However, if $y_0 \in V \setminus D(A)$, then we consider $y(t) = S(t)y_0$ as a generalized solution of (0.2.1). Also, one can easily verify that the solution $y(t) = S(t)y_0$, $t \geq 0$ is unique.*

Theorem 0.2.7. *If two C_0-semigroups have the same infinitesimal generator, then they are identical.*

Remark 0.2.8. *If A is bounded operator on some Banach space V, then $S(t) = e^{tA}, t \geq 0$ is a C_0-semigroup whose infinitesimal generator is A. By above theorem, we can conclude that the converse is also true, i.e., if $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup with bounded infinitesimal generator A then $S(t) = e^{tA}$.*

Now, the question of existence of unique solution of system (0.2.1) has been reduced to the question whether the operator A is infinitesimal generator of some semigroup or not. We now state few results giving necessary and sufficient conditions for an operator A to be infinitesimal generator of some C_0-semigroup.

Theorem 0.2.9 (Hille-Yosida theorem)**.** *Let V be a Banach space. A linear unbounded operator on V is infinitesimal generator of a contraction semigroup iff A satisfies:*

(i) A is closed and densely defined, and

(ii) $(\lambda I - A)^{-1}$ is a bounded linear operator for all $\lambda > 0$ satisfying $||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}$.

The above theorem can be generalized for any C_0-semigroup (see [4]). We have easier conditions to be checked in case of Hilbert spaces, which will be stated next.

Definition 0.2.10 (Symmetric and Self-adjoint operators)**.** *Let H be any Hilbert space and let $A : D(A) \subset H \to H$ be a densely defined linear operator. Then we say A is symmetric if*

$$\langle Aw_1, w_2 \rangle_H = \langle w_1, Aw_2 \rangle_H \,, \quad \forall w_1, w_2 \in D(A).$$

If $D(A) = H$, then $A^* = A$ and A is called self-adjoint.

Definition 0.2.11 (Maximal dissipative operators)**.** *Let H be a Hilbert space. A linear operator $A : D(A) \subset H \to H$ is said to be dissipative if $\langle Aw, w \rangle_H \leq 0$ for every $w \in H$ and maximal dissipative if it is dissipative and $R(I + A) = H$.*

In the field of control theory, we work in Hilbert spaces, in general. So let us see the sufficient and necessary conditions (which are easier to be checked) for an operator A to be infinitesimal generator.

Theorem 0.2.12. *Let H be a Hilbert space. Then a linear operator $A : D(A) \subset H \to H$ is infinitesimal generator of some C_0-semigroup iff it is maximal dissipative.*

Theorem 0.2.13. *Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a linear operator such that A and $-A$ are maximal dissipative operators, then they together generate a group of isometries.*

Now, let us consider the inhomogeneous equation:

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + Bq(t), \quad t \in (0, T), \\ y(0) = y_0 \end{cases} \tag{0.2.2}$$

where, A is an infinitesimal generator of some C_0-semigroup $\{S(t)\}_{t\geq 0}$, defined on a vector space V, and $q : [0, T] \to V$ is a given map.

Definition 0.2.14 (Classical solution of (0.2.2))**.** *A map $y : [0, T] \to V$ is said to be a classical solution of (0.2.2) if y is continuous on $[0, T]$, continuously differentiable on $(0, T)$, $u(t) \in D(A)$ for $t \in (0, T)$, and satisfies (0.2.2) on $(0, T)$.*

Remark 0.2.15. *If y is a classical solution of (0.2.2), then y will be of the form:*

$$y(t) = S(t)y_0 + \int_0^t S(t-s)q(s)\,ds \tag{0.2.3}$$

This expression for the solution of (0.2.3) is analogous to the expression (0.1.2), which is the solution of inhomogeneous ODE, (0.1.1).

Note that if we assume q to be integrable, then the above expression for solution y is sensible, and so we call (0.2.3) to be a generalized solution.

Theorem 0.2.16. *The generalized solution y, given by (0.2.3) is classical for any $y_0 \in D(A)$ iff $y(t)$ is continuously differentiable on $(0,T)$, equivalently, $y(t) \in D(A)$ for $t \in (0,T)$ and $Ay(t)$ is continuous on $(0,T)$.*

Theorem 0.2.17. *If $q \in C^1([0,T;V])$, then the equation (0.2.2) has a unique classical solution for every $y_0 \in D(A)$.*

0.3 Infinite Dimensional Control Systems (PDEs)

In this section, we will describe the controllability of infinite-dimensional system in abstract setting. Let H and U be infinite dimensional Hilbert spaces. In this section, H and U denotes the state space and control space, respectively. Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup of continuous linear operators on H and let A be its infinitesimal generator. We will denote the denote the adjoint of $S(t)$ by $S(t)^*$. Then $\{S(t)^*_{t\geq 0}\}$ is also a C_0-semigroup with A^* (adjoint of A) as the corresponding infinitesimal generator. Note that $D(A^*)$ equipped with the inner product

$$\langle w_1, w_2\rangle_{D(A^*)} := \langle w_1, w_2\rangle_H + \langle A^* w_1, A^* w_2\rangle_H\,, \quad \forall (w_1, w_2) \in D(A^*)^2$$

defines it as a Hilbert space. We denote the dual of $D(A^*)$ by $D(A^*)'$ with respect to the pivot space H, in particular, we have $D(A^*) \subset H \subset D(A^*)'$.

Let $B \in \mathcal{L}(U, D(A^*))$, i.e., B is a linear map from U into the space of linear maps from $D(A^*)$ into $\mathbb{R}$ such that for some $C > 0$, we have:

$$|(Bq)w| \leq C||q||_U||w||_{D(A^*)},\ \forall q \in U, \forall w \in D(A^*).$$

We also assume

$$\forall T > 0,\, \exists C_T > 0 : \int_0^T ||B^* S(t)^* w||_U^2\,dt \leq C_T||w||_H^2,\ \forall w \in D(A^*). \tag{0.3.1}$$

We call this assumption as 'admissibility condition'. Using this assumption one can easily conclude that the operators:

$$(w \in D(A^*)) \mapsto ((t \mapsto B^* S(t)^* w) \in C([0,T];U))\,,$$
$$(w \in D(A^*)) \mapsto ((t \mapsto B^* S(T-t)^* w) \in C([0,T];U))$$

can be extended uniquely as continuous linear maps from H into $L^2(0,T;U)$. In this section we will consider these extended maps and by abuse of notation, we will use the same notation to represent the new extended maps.

Claim 0.3.1. *The admissibility condition* (0.3.1) *is equivalent to*

$$\exists\, T>0,\, \exists\, C_T>0: \int_0^T ||B^*S(t)^*w||_U^2\,dt \le C_T||w||_H^2,\ \forall w\in D(A^*). \tag{0.3.2}$$

Proof of the claim Obviously, (0.3.1) implies (0.3.2). Now, assume (0.3.2) is true for some $T=T_0$. Let $T>0$ be arbitrary.

$T\le T_0:$ In this case, $\int_0^{T_0} ||B^*S(t)^*w||_U^2\,dt \le \int_0^T ||B^*S(t)^*w||_U^2\,dt \le C_T||w||_H^2,\ \forall w\in D(A^*)$.

$T\ge T_0$: For any $w\in D(A^*)$, we have

$$\begin{aligned}
\int_0^{2T_0} ||B^*S(t)^*w||_U^2\,dt &= \int_0^{T_0} ||B^*S(t)^*w||_U^2\,dt + \int_{T_0}^{2T_0} ||B^*S(t)^*w||_U^2\,dt \\
&= \int_0^{T_0} ||B^*S(t)^*w||_U^2\,dt + \int_0^{T_0} ||B^*S(t+T_0)^*w||_U^2\,dt \\
&\le \int_0^{T_0} ||B^*S(t)^*w||_U^2\,dt + ||S(T_0)^*||^2 \int_{T_0}^{T_0} ||B^*S(t)^*w||_U^2\,dt \\
&\le \widetilde{C}_{T_0}||w||_H^2,\ \text{for some } \widetilde{C}_{T_0}>0.
\end{aligned}$$

For any given $T>0$, there exist some $n\in\mathbb{N}$ such that $nT_0\le T<(n+1)T_0$, so we have:

$$\begin{aligned}
\int_0^{T} ||B^*S(t)^*w||_U^2\,dt &= \int_0^{nT_0} ||B^*S(t)^*w||_U^2\,dt + \int_0^{T-nT_0} ||B^*S(t+nT_0)^*w||_U^2\,dt \\
&\le \overline{C}_{T_0}||w||_H^2,\ \text{for some } \overline{C}_{T_0}>0.
\end{aligned}$$

□

Let $T>0$. Consider the following control system:

$$\begin{cases} \dfrac{dy}{dt} = Ay+Bq, \quad t\in(0,T) \\ y(0)=y_0. \end{cases} \tag{0.3.3}$$

Definition 0.3.2 (Solution of (0.3.3))**.** *Let $y_0\in H, q\in L^2(0,T;U)$. We say a function $y\in C([0,T];H)$ is a solution to the system* (0.3.3) *if $\forall\tau\in[0,T]$ and $\forall\varphi_\tau\in H$, it satisfies*

$$\langle y(\tau),\varphi_\tau\rangle_H - \langle y_0, S(\tau)^*\varphi_\tau\rangle_H = \int_0^\tau \langle q(t), B^*S(\tau-t)^*\varphi_\tau\rangle_U\,dt$$

Note that the right hand side term in the above equation is well defined due to the assumption of admissibility condition (0.3.1).

Theorem 0.3.3 (Well posedness)**.** *For any given initial data $y_0\in H$ and any given control function $q\in L^2(0,T;U)$, the system* (0.3.3) *has a unique solution y. Moreover, there exists $C>0$, depending on T but independent of y_0, q, such that*

$$||y(\tau)||_H \le C\left(||y_0||_H + ||q||_{L^2(0,T;U)}\right)$$

Now, we can define the different notions of controllability for the infinite dimensional system (0.3.3).

Definition 0.3.4 (Exact Controllability). *System (0.3.3) is said to be exactly controllable in some given time $T > 0$, if for any $y_0, y_1 \in H$ there exists $q \in L^2(0, T; U)$ such that the solution of (0.3.3) satisfies $y(T) = y_1$.*

Definition 0.3.5 (Approximate Controllability). *System (0.3.3) is said to be approximately controllable in some given time $T > 0$, if for any $y_0, y_1 \in H$ and for any $\epsilon > 0$ there exists $q_\epsilon \in L^2(0, T; U)$ such that the solution of (0.3.3) satisfies $|y(T) - y_1| < \epsilon$.*

Definition 0.3.6 (Null Controllability). *System (0.1.1) is said to be null controllable in some given time $T > 0$, if for any $y_0 \in H$, there exists $q \in L^2(0, T; U)$ such that the solution of (0.3.3) satisfies $y(T) = 0$.*

Remark 0.3.7. *Let us now explore the relation between these notions of controllability, as done in Remark 0.1.7.*

- *It is obvious from the definition that exact controllability of the system (0.3.3) implies its null controllability and approximate controllability as well.*
- *The converse of the above statement is not true, in general as in case of finite dimensional systems. For example, heat equation $y_t - y_{xx} = 0$ is null controllable and approximate controllable in proper space settings but not exactly controllable due to smoothing effects.*
- *Assume $\{\mathbf{S(t)}\}_{\mathbf{t}\in\mathbb{R}}$ **forms a $\mathbf{C_0}$-group** of linear operators, whose infinitesimal generator is A. In this situation, the converse statement is true. More precisely, if the system (0.3.3) is null controllable in some time $T > 0$, then the system is exactly controllable in time T (and hence approximately controllable).*

It is now clear that the different notions of controllability mentioned above are not equivalent, in general, as in case of finite dimensional system. So, the duality approach to study the different notions of controllability are different and can be stated as follows:

Theorem 0.3.8 (Exact Controllability). *The control system (0.3.3) is exactly controllable in some time $T > 0$ iff there exist some $C > 0$ such that*

$$\int_0^T ||B^* S(t)^* \varphi_T||_U^2 \, dt \geq C||\varphi_T||_H^2, \ \forall \varphi_T \in D(A^*) \quad \text{[Observability Inequality]}$$

Theorem 0.3.9 (Null Controllability). *The control system (0.3.3) is null controllable in some time $T > 0$ iff there exist some $C > 0$ such that*

$$\int_0^T ||B^* S(t)^* \varphi_T||_U^2 \, dt \geq C||S(T)^* \varphi_T||_H^2, \ \forall \varphi_T \in D(A^*) \quad \text{[Weak Observability Inequality]}$$

Theorem 0.3.10 (Approximate Controllability). *The control system (0.3.3) is approximately controllable in some time $T > 0$ iff for every $\varphi_T \in H$,*

$$B^* S(\cdot)^* \varphi_T = 0 \text{ in } L^2(0, T; U) \implies \varphi_T = 0. \quad \text{[Unique Continuation Principle]}$$

Remark 0.3.11. *Note that the Observability inequality implies the Weak observability inequality and Unique continuation principle, but the converse is not true in general, which matches with the points mentioned in Remark 0.3.7.*

0.4 Methods to deal with controllability

From the above sections it is quite clear that the problem of controllability of any system can be done by two approaches mentioned below:

- Direct approach, which involves finding of explicit form od the control and not just existence.
- Dual approach, which involves proving of the equivalent criteria, based on the adjoint system.

Many methods have been developed so far to deal with the controllability of a system. Few methods among them are listed below:

1. Method of moments
2. Flatness method
3. Backstepping ...

} Direct methods

4. Carleman estimates
5. Labeau-Robianno's startegy
6. Transmutation method ...

} Duality methods

In the Chapter 2, Carleman estimates has been used to deal with exact, while in Chapter 2 and Chapter 3, the method of moments has been employed for proving null controllability of the concerned systems.

CHAPTER 1

Introduction

It is quite clear from the title of the thesis that we will try to control something, more precisely, some systems described by differential equations. In usual sense, controlling someone or something means to have influence on it. In the same line of understanding, controllability of an evolutionary differential equation describes whether one can influence its dynamics at some given time $T > 0$ using some input function, called control, and get the desired behavior or not. The notion of controllability for the differential equation will be made precise in the next Chapter 2.

In this thesis, we will study the controllability aspect of the following two types of partial differential equations, posed in $(0, T) \times (0, 2\pi)$ with periodic boundary conditions:

1. Transport Equation (Hyperbolic PDE)

$$\begin{cases} \eta_t + a\eta_x = 0, & (t, x) \in (0, T) \times (0, 2\pi), \\ \eta(t, 0) = \eta(t, 2\pi), & t \in (0, T), \\ \eta(0, x) = \eta_0(x), & x \in (0, 2\pi), \end{cases} \tag{1.0.1}$$

   where $a \in \mathbb{R} \setminus \{0\}$ is the velocity/speed.

2. Kuramoto-Sivashinsky-Korteweg-de Vries Equation (Parabolic PDE)

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + \nu u_{xx} = 0, & (t, x) \in (0, T) \times (0, 2\pi), \\ u(t, 0) = u(t, 2\pi), u_x(t, 0) = u_x(t, 2\pi), & t \in (0, T), \\ u_{xx}(t, 0) = u_{xx}(t, 2\pi), u_{xxx}(t, 0) = u_{xxx}(t, 2\pi), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 2\pi), \end{cases} \tag{1.0.2}$$

   where the coefficients $\gamma, \nu > 0$ accounts for the long-wave instabilities and the short wave dissipation respectively.

The Kuramoto- Sivashinsky (KS) equation, $u_t + \gamma u_{xxxx} + \nu u_{xx} + uu_x = 0$ was first proposed independently by Kuramoto and Tsuzuki as model for Beluozov-Zabotinskii reaction patterns in reaction-diffusion system in [5] and by Sivashinsky as model for unstable flame fronts in [6]. Later this equation has been studied in a series of papers [7], [8], [9], [10], [11], [12], [13] and references therein. The following KS-KdV equation was introduced by Benney in [14] to include dispersive effects by adding the KdV term u_{xxx}. The following KS-KdV equation derived by Benney in [14]

$$u_t + \gamma u_{xxxx} + u_{xxx} + \nu u_{xx} + u u_x = 0 \tag{1.0.3}$$

is used to study a wide range of nonlinear dissipative waves. The study of controllability aspect of parabolic partial differential equations has gained a lot of interest among researchers throughout the years. As a result, many methods have been developed and employed to study the controllability of parabolic PDEs, in particular for heat equation, like Moment method (see [15]), Transmutation method (see [16]), Flatness method (see [17]), Backstepping approach (see [18]) and Carleman estimates (see [19], [20]). Among these, the following two methods have been mainly explored for the controllability of the KS equation $u_t+\gamma u_{xxxx}+\nu u_{xx}+uu_x = 0$ and/or linear KS equation $u_t + \gamma u_{xxxx} + \nu u_{xx} = 0$ so far:

- Moment method (see [21], [22], etc)
- Carleman estimates (see [23], [22], [24], [25], etc).

Controllability of hyperbolic partial differential equation has also been extensively studied. Besides moment method [26] and Carleman approach [27], multiplier method [28] is very useful for this type of model.

In this thesis, we prove the exact controllability of transport equation (1.0.1) in $L^2(0, 2\pi)$ at time $T > \frac{2\pi}{|a|}$ by means of localized interior and boundary control, using the Carleman estimates. Further, we also prove the negative result of null (and hence exact) controllability for time $T < \frac{2\pi}{|a|}$ using the characteristic method. For, the KS-KdV equation, we prove the null controllability in $(H^2(0, 2\pi))^*$ by means of localized interior and boundary control at any time $T > 0$, using the method of moments.

CHAPTER 2

Controllability of Transport Equation

2.1 Introduction

This chapter is concerned with the study of exact controllability of the simplest hyperbolic equation, transport equation, described by (1.0.1), but posed in $(0,T)\times(0,2\pi)$, where $T>0$. More precisely, we will prove the (localized) interior controllability and boundary controllability using the duality approach by proving the corresponding observability inequality. We use the Carleman estimates to prove the observability inequalities in both cases. Let us first write the concerned control systems. The control system concerning the interior controllability can be written as:

$$\begin{cases} \eta_t + a\eta_x = \mathbb{1}_\omega h, & (t,x)\in(0,T)\times(0,2\pi), \\ \eta(t,0)=\eta(t,2\pi), & t\in(0,T), \\ \eta(0,x)=\eta_0(x), & x\in(0,2\pi), \end{cases} \tag{2.1.1}$$

where $\omega=(0,\xi)\cup(2\pi-\xi,2\pi)$ is an open subset of $(0,2\pi)$ and $h=h(t,x)$ is the interior control function with localized support in ω. The underlying space operator for this equation is given by

$$Aw=-aw_x, \quad \forall w\in D(A)=H^1_p(0,2\pi),$$

and the operator B is given by

$$B: L^2(0,T;L^2(0,2\pi))\to D(A^*)' : Bw=\mathbb{1}_\omega w$$

Proposition 2.1.1 (Well-Posedness of (2.1.1))**.** *For any given $h\in L^2(0,T;L^2(\omega))$ and $\eta_0\in L^2(0,2\pi)$, equation (2.1.1) has a unique solution $\eta\in C([0,T];L^2(0,2\pi))$ given by the variation of parameters formula as*

$$\eta(t)=S(t)\eta_0+\int_0^t S(t-s)\mathbb{1}_\omega h(s)\,ds$$

where, $S(t)$ is the C_0-semigroup corresponding to the operator A.

Proof. One can easily prove A to be maximal dissipative, and then the result follows from the standard theory of Inhomogeneous equation (see 0.2.3).

□

The boundary control system can be written as:

$$\begin{cases} \eta_t + a\eta_x = 0, & (t,x) \in (0,\infty) \times (0,2\pi), \\ \eta(t,0) = \eta(t,2\pi) + q(t), & t \in (0,\infty), \\ \eta(0,x) = \eta_0(x), & x \in (0,2\pi), \end{cases} \tag{2.1.2}$$

where q is boundary control.

Note that, $Aw = -aw_x$ for $w \in D(A) = H_p^1(0,2\pi)$, and so $A^*w = aw_x$ with $w \in D(A^*) = (H_p^1)'(0,2\pi)$, where $(H_p^1)'(0,2\pi)$ is dual of $H_p^1(0,2\pi)$ with respect to the pivot space $L^2(0,2\pi)$. Thus, the adjoint system of (2.1.1) is given by:

$$\begin{cases} \varphi_t + a\varphi_x = 0, & (t,x) \in (0,T) \times (0,2\pi), \\ \varphi(t,0) = \varphi(t,2\pi), & t \in (0,T), \\ \varphi(T,x) = \varphi_T(x), & x \in (0,2\pi). \end{cases} \tag{2.1.3}$$

Let us now give the definition of solution for the boundary control system (2.1.3).

Definition 2.1.2 (Solution of (2.1.2))**.** *Let $T > 0$, $\eta_0 \in L^2(0,2\pi)$ and $q \in L^2(0,T)$ be given. We say $\eta \in C([0,T]; L^2(0,2\pi))$ is a solution of the system (2.1.2), if for every $\tau \in [0,T]$, and for every $\varphi \in C^1([0,\tau] \times [0,2\pi])$ with $\varphi(t,0) = \varphi(t,2\pi)$ satisfies:*

$$-\int_0^{2\pi}\int_0^{\tau} (\varphi_t + \varphi_x)\eta \, dtdx + \int_0^{2\pi} \eta(\tau,x)\varphi(\tau,x)\, dx - \int_0^{2\pi} \eta_0(x)\varphi(0,x)\, dx = a\int_0^{2\pi} q(t)\varphi(t,0)\, dt \tag{2.1.4}$$

Proposition 2.1.3 (Well-Posedness of (2.1.2))**.** *For any given $q \in L^2(0,T)$ and any initial data $\eta_0 \in L^2(0,2\pi)$, the system (2.1.2) has a unique solution $\eta \in C([0,T]; L^2(0,2\pi))$ (in sense of (2.1.2)).*

Proof. This follows from the semigroup theory (see Remark 0.2.6) as A is maximal dissipative.

□

2.2 Interior Controllability

In this section, we will study the null and exact controllability of the transport equation (1.0.1) with control acting through the interior of domain. More precisely, we will consider the system (2.1.1). As mentioned earlier, our study will be based on the duality approach and so we need to find the equivalent observability inequality for null and exact controllability. In this case, we consider the space $L^2(0,T; L^2(0,2\pi))$ as the state and control space. Then the operator $B : L^2(0,T; L^2(0,2\pi)) \to D(A^*)'$ is given by $Bw = \mathbb{1}_\omega w$ and so $B^* : D(A^*) \to L^2(0,T; L^2(0,2\pi))$ is given by $B^*w = \mathbb{1}_\omega w$. Thus, the equivalent observability inequality for null controllability and exact controllability respectively is given as:

$$||S(T)^*\varphi_T||_{L^2(0,2\pi)} \le C_1||\varphi||_{L^2(0,T;L^2(\omega))} \text{ and } ||\varphi_T||_{L^2(0,2\pi)} \le C_2||\varphi||_{L^2(0,T;L^2(\omega))}, \forall\, \varphi_T \in D(A^*). \tag{2.2.1}$$

where $C_1, C_2 > 0$ are some constants, and $S(T)^*\varphi_T = \varphi(0,\cdot)$. Let us mention the controllability result for the transport equation (1.0.1) precisely.

Theorem 2.2.1 (Exact Controllability). *Let $T > \frac{2\pi}{|a|}$. Then for any $\eta_0, \eta_1 \in L^2(0, 2\pi)$, there exists an interior control $h \in L^2(0,T; L^2(0, 2\pi))$ supported in an open set $\omega = (0, \xi) \cup (2\pi - \xi, 2\pi)$ for some $\xi > 0$ such that the solution of system (2.1.1) satisfies $\eta(T, \cdot) = \eta_1$.*

Let us first derive the corresponding Carleman estimate, from where the desired observability inequality would be immediate. W.L.O.G, we assume $a > 0$ for proving the Carleman estimate. Assume $T > \frac{2\pi}{a}$, and so choose $\delta > 0$ and $\rho \in (0, 1)$ such that

$$\rho\, aT > 2\pi + \delta. \tag{2.2.2}$$

Define a function $\psi \in C^\infty([0, 2\pi])$ such that

$$\psi(x) = |x + \delta|^2 \text{ for } x \in [\xi/2, 2\pi - \xi/2], \tag{2.2.3}$$

$$\frac{d\psi}{dx}(0) = \frac{d\psi}{dx}(2\pi), \tag{2.2.4}$$

$$2\delta \leq \frac{d\psi}{dx}(x) \leq 2(2\pi + \delta) \text{ for } x \in [0, 2\pi]. \tag{2.2.5}$$

We now define the weight function $\alpha \in C^\infty([0, 2\pi] \times \mathbb{R})$ as

$$\alpha(x, t) = \psi(x) - \rho a^2 t^2. \tag{2.2.6}$$

Lemma 2.2.2 (Carleman estimate (borrowed from [29])). *Let ω, T and α be as described above. Then there exists some positive constant C such that for all $\varphi \in L^2(0,T; L^2(0, 2\pi))$ with $\varphi_t + a\varphi_x \in L^2(0,T; L^2(0, 2\pi))$, we have:*

$$\int_0^T \int_0^{2\pi} |\varphi|^2 e^{2\alpha}\, dx\, dt + \int_0^{2\pi} |\varphi(0, x)|^2 e^{2\alpha(0,x)}\, dx + \int_0^{2\pi} |\varphi(T, x)|^2 e^{2\alpha(T,x)}\, dx$$
$$\leq C_1 \left(\int_0^T \int_0^{2\pi} |\varphi_t + a\varphi_x|^2 e^{2\alpha}\, dx\, dt + \int_0^T \int_\omega |\varphi|^2 e^{2\alpha}\, dx\, dt \right). \tag{2.2.7}$$

Proof. We first assume that $\varphi \in H^1((0,T) \times (0, 2\pi))$ with $\varphi(t, 0) = \varphi(t, 2\pi)$. Let $v = e^\alpha \varphi$ and $D = \partial_t + a\,\partial_x$. Then

$$\begin{aligned} e^\alpha D\varphi &= e^\alpha D(e^{-\alpha} v) \\ &= (-\alpha_t v - a\alpha_x v) + (v_t + av_x) \\ &=: D_1 v + D_2 v \end{aligned}$$

$$\|e^\alpha D\varphi\|^2_{L^2((0,T)\times(0,2\pi))} = \|D_1 v\|^2_{L^2((0,T)\times(0,2\pi))} + \|D_2 v\|^2_{L^2((0,T)\times(0,2\pi))} + 2(D_1 v, D_2 v)_{L^2((0,T)\times(0,2\pi))} \tag{2.2.8}$$

Using integration by parts, we get:

$$2(D_1 v, D_2 v)_{L^2((0,T)\times\mathbb{T})} = \int_0^T \int_0^{2\pi} (\alpha_{tt} + 2a\alpha_{tx} + a^2\alpha_{xx}) v^2\, dx\, dt$$
$$- \int_0^{2\pi} (\alpha_t + a\alpha_x) v^2|_0^T\, dx - \int_0^T a(\alpha_t + a\alpha_x) v^2|_0^{2\pi}\, dt \tag{2.2.9}$$

Using (2.2.4), (2.2.6) and the fact that $v(t, 0) = v(t, 2\pi)$, the last integral becomes 0. Also, from (2.2.2)-(2.2.6), we have:

for $(t,x) \in (0,T) \times (\eta/2, 2\pi - \eta/2) \supset (0,T) \times \omega^c$

$$\begin{aligned}\alpha_{tt} + 2a\alpha_{xt} + a^2\alpha_{xx} &= -2\rho a^2 + 2a.0 + a^2\psi_{xx} \\ &= 2a^2(1-\rho) > 0,\end{aligned}$$

for $(t,x) \in \{T\} \times (0,2\pi)$

$$\begin{aligned}-(\alpha_t + a\alpha_x) &= 2\rho a^2 T - a\psi_x \\ &\geq 2\rho a^2 T - 2a(2\pi + \delta) \\ &= 2a(\rho a T - (2\pi + \delta)) > 0,\end{aligned}$$

and for $(x,t) \in \{0\} \times (0,2\pi)$

$$\alpha_t + c\alpha_x = 0 + c\psi_x > c2\delta > 0.$$

Also as $\alpha \in C^\infty((0,T) \times (0,2\pi))$, so for $(t,x) \in (0,T) \times (0,2\pi) \supset (0,T) \times \omega$, we have:

$$\alpha_{tt} + 2a\alpha_{xt} + a^2\alpha_{xx} \geq -K_1, \text{ for some } K_1 > 0.$$

Using the above facts, we obtain:

$$\begin{aligned}&\|e^\alpha D\varphi\|^2{}_{L^2((0,T)\times(0,2\pi))} \\ &\quad = \|D_1 v\|^2_{L^2((0,T)\times(0,2\pi))} + \|D_2 v\|^2_{L^2((0,T)\times(0,2\pi))} + 2(D_1 v, D_2 v)_{L^2((0,T)\times(0,2\pi))} \\ &\quad = \|D_1 v\|^2_{L^2((0,T)\times(0,2\pi))} + \|D_2 v\|^2_{L^2((0,T)\times(0,2\pi))} + \int_0^T \int_{0)^{2\pi}} (\alpha_{tt} + 2a\alpha_{tx} + a^2\alpha_{xx}) v^2 \, dx \, dt \\ &\qquad - \int_0^{2\pi} (\alpha_t + a\alpha_x) v^2 |_0^T \, dx \\ &\quad \geq \int_0^T \int_\omega (\alpha_{tt} + 2a\alpha_{tx} + a^2\alpha_{xx}) v^2 \, dx \, dt + \int_0^T \int_{\omega^c} K_1 v^2 \, dx \, dt \\ &\qquad + \int_0^{2\pi} K_2 \left(|v(0,x)|^2 + |v(T,x)|^2 \right) dx \\ &\quad \geq -\int_0^T \int_\omega K_3 v^2 \, dx \, dt + \int_0^T \int_{\omega^c} K_1 v^2 \, dx \, dt + \int_0^{2\pi} K_2 \left(|v(0,x)|^2 + |v(T,x)|^2 \right) dx\end{aligned}$$

On adding $\int_0^T \int_\omega K_1 v^2 \, dx \, dt$ both sides, we get:

$$\begin{aligned}&\|e^\alpha D\varphi\|^2_{L^2((0,T)\times(0,2\pi))} + \int_0^T \int_\omega K_1 v^2 \, dx \, dt \\ &\geq -\int_0^T \int_\omega K_3 v^2 \, dx \, dt + \int_0^T \int_0^{2\pi} K_1 v^2 \, dx \, dt + \int_0^{2\pi} K_2 \left(|v(0,x)|^2 + |v(T,x)|^2 \right) \, dx\end{aligned}$$

which implies:

$$\begin{aligned}&\|e^\alpha D\varphi\|^2_{L^2((0,T)\times(0,2\pi))} + \int_0^T \int_\omega (K1 + K_3) v^2 \, dx \, dt \\ &\qquad \geq \int_0^T \int_0^{2\pi} K_1 v^2 \, dx \, dt + \int_0^{2\pi} K_2 \left(|v(0,x)|^2 + |v(T,x)|^2 \right) \, dx\end{aligned}$$

Thus, for some $C_1 > 0$ we have:

$$\int_0^T \int_0^{2\pi} |v|^2 \, dx \, dt + \int_0^{2\pi} \left(|v(0,x)|^2 + |v(T,x)|^2\right) \, dx$$
$$\leq C_1 \left(\int_0^T \int_0^{2\pi} |e^{\alpha} D\varphi|^2 \, dx \, dt + \int_0^T \int_\omega |v|^2 \, dx \, dt \right)$$

Substituting v by $e^{\alpha}\varphi$, we get (2.2.7).

We now prove that the above lemma is still true when φ and $f := D\varphi$ are in $L^2(0,T; L^2((0,2\pi)))$ using sequential argument. Indeed, in such case $\varphi \in C([0,T]; L^2((0,2\pi)))$, and if (φ_0^n) and (f^n) are two sequences in $H^1(0,2\pi)$ and $L^2(0,T; H^1(0,2\pi))$, respectively, such that

$$\varphi_0^n \to \varphi(0,\cdot) \text{ in } L^2((0,2\pi))$$
$$f^n \to f \text{ in } L^2(0,T; L^2((0,2\pi)))$$

then the solution $\varphi^n \in C([0,T]; H^1(0,2\pi))$ of

$$\varphi_t^n + a\,\varphi_x^n = f^n$$
$$\varphi^n(0) = \varphi_0^n$$

satisfies $\varphi^n \in H^1((0,T) \times (0,2\pi))$ and $\varphi^n \to \varphi$ in $C([0,T]; L^2(0,2\pi))$, so that we can apply the above lemma to φ^n and then pass to the limit as $n \to \infty$ in the Carleman estimate for φ^n to get the estimate for φ.

□

Proof of Theorem 2.2.1 We know that proving this theorem is equivalent to proving the corresponding observability inequality, mentioned in (2.2.1).

Now, substituting φ as the solution of the adjoint system (2.1.3), we get:

$$\int_0^T \int_0^{2\pi} |\varphi|^2 + \int_0^{2\pi} |\varphi(0,x)|^2 + \int_0^{2\pi} |\varphi(T,x)|^2 \leq \int_0^T \int_\omega |\varphi|^2, \tag{2.2.10}$$

where we have bounded the exponential functions as $\alpha \in C^\infty((0,T) \times (0,2\pi))$. As all the terms on the left hand side are positive, we obtain:

$$\int_0^{2\pi} |\varphi_T(x)|^2 \leq \int_0^T \int_\omega |\varphi|^2$$

which proves the theorem.

□

Remark 2.2.3. *We know that exact controllability implies null controllability, in particular. So, the transport equation is null controllable in time $T > \frac{2\pi}{|a|}$ by means of localized interior control, i.e., the solution of system* (2.1.1) *vanishes at time $t = T$.*

One can also conclude this directly from (2.2.10)*. Since all the terms on the left hand side of* (2.2.10) *are positive, so we have:*

$$\int_0^{2\pi} |\varphi(0,x)|^2 \leq \int_0^T \int_\omega |\varphi|^2$$

which proves null controllability (see (2.2.1)*).*

2.3 Boundary Controllability

In this section, we will deal with the boundary controllability of the transport equation (1.0.1). So, let us consider the system (2.1.2). We consider the spaces $L^2(0,T;L^2(0,2\pi))$ and $\mathbb{R}$ as the state space and the control space respectively, for this system. Then, comparing the Definition 2.1.2 with Definition 0.3.2 the operator $B^* : D(A^*) \to \mathbb{R}$ can be given as $B^*w = w(t,0)$, and so the observability inequality for null and exact controllability can be given as:

$$||S(T)^*\varphi_T||_{L^2(0,2\pi)} \leq C_1||\varphi(t,0)||_{L^2(0,T)} \text{ and } ||\varphi_T||_{L^2(0,2\pi)} \leq C_2||\varphi(t,0)||_{L^2(0,T)}, \forall\, \varphi_T \in D(A^*). \tag{2.3.1}$$

where $C_1, C_2 > 0$ are some constants and $S(T)^*\varphi_T(x) = \varphi(0,x)$. Now, let us state the theorems concerning null and exact boundary controllability of the transport equation, (1.0.1).

Theorem 2.3.1 (Exact Controllability). *Let $T > \frac{2\pi}{|a|}$. Then for any $\eta_0, \eta_1 \in L^2(0,2\pi)$, there exists a boundary control $q \in L^2(0,T)$ such that the solution of system (2.1.2) satisfies $\eta(T,\cdot) = \eta_1$.*

Again, we will derive a Carleman inequality from where the proof of observability inequality would be trivial. For the derivation of Carleman inequality, we assume without loss of generality that $a > 0$. Let $T > \frac{2\pi}{a}$, then we can choose $\delta > 0$ and $\rho \in (0,1)$ such that

$$\rho\, aT > 2\pi + \delta. \tag{2.3.2}$$

Let us define a function as

$$\widetilde{\psi}(x) = |x+\delta|^2, \forall x \in [0,2\pi]. \tag{2.3.3}$$

We next define the weight function $\widetilde{\alpha} \in C^\infty((0,T)\times[0,2\pi]))$ as

$$\widetilde{\alpha}(x,t) = \widetilde{\psi}(x) - \rho a^2 t^2. \tag{2.3.4}$$

Then, we have the following lemma:

Lemma 2.3.2 (Carleman estimate). *Let T and $\widetilde{\alpha}$ be as described above. Then there exists some positive constant $\widetilde{C}$ such that for all $\varphi \in L^2(0,T;L^2(0,2\pi))$ with $\varphi_t + a\varphi_x \in L^2(0,T;L^2(0,2\pi))$, we have:*

$$\int_0^T\int_0^{2\pi} |\varphi|^2\, e^{2\alpha}\, dx\, dt + \int_0^{2\pi} |\varphi(0,x)|^2\, e^{2\alpha(0,x)}\, dx + \int_0^{2\pi} |\varphi(T,x)|^2\, e^{2\alpha(T,x)}\, dx$$
$$\leq C_1 \left(\int_0^T\int_0^{2\pi} |\varphi_t + a\varphi_x|^2\, e^{2\alpha}\, dx\, dt + \int_0^T \left(e^{2\alpha(t,2\pi)} + e^{2\alpha(t,0)}\right) |\varphi(t,0)|^2\, dt \right). \tag{2.3.5}$$

Proof. Following the same notations and proof of Lemma 2.2.2, we have:

$$\begin{aligned}
&\|e^{\alpha}D\varphi\|^2{}_{L^2((0,T)\times(0,2\pi))}\\
&= \|D_1 v\|^2_{L^2((0,T)\times(0,2\pi))} + \|D_2 v\|^2_{L^2((0,T)\times(0,2\pi))} + \int_0^T\int_0^{2\pi}(\alpha_{tt}+2a\alpha_{tx}+a^2\alpha_{xx})v^2\,dx\,dt\\
&\qquad - \int_0^{2\pi}(\alpha_t+a\alpha_x)v^2|_{t=0}^{T}\,dx + \int_0^T\left[(\alpha_t+a\alpha_x)|v|^2\right]_{x=0}^{2\pi}dt\\
&\geq \int_0^T\int_0^{2\pi}(\alpha_{tt}+2a\alpha_{tx}+a^2\alpha_{xx})v^2\,dx\,dt - \int_0^{2\pi}(\alpha_t+a\alpha_x)|_{t=T}|v(T,x)|^2\,dx\\
&\qquad + \int_0^{2\pi}(\alpha_t+a\alpha_x)|_{t=0}|v(0,x)|^2\,dx + \int_0^T(\alpha_t+a\alpha_x)|_{x=2\pi}|v(t,2\pi)|^2\,dt\\
&\qquad - \int_0^T(\alpha_t+a\alpha_x)|_{x=0}|v(t,0)|^2\,dt\\
&:= I_1+I_2+I_3+I_4+I_5
\end{aligned} \tag{2.3.6}$$

Now, we have:

$$\begin{aligned}
I_1 &= \int_0^T\int_0^{2\pi}\left(-2a^2\rho+2a^2\right)v^2\,dx\,dt = \underbrace{2a^2(1-\rho)}_{>0}\int_0^T\int_0^{2\pi}|v(t,x)|^2\,dt\,dx\\
I_2 &= -\int_0^{2\pi}2a\left(-\rho aT+x+\delta\right)|v(T,x)|^2 \geq \underbrace{2a(\rho aT-2\pi-\delta)}_{>0}\int_0^{2\pi}|v(T,x)|^2\,dx\\
I_3 &= 2a\int_0^{2\pi}(x+\delta)\,|v(0,x)|^2\,dx \geq \underbrace{2a\delta}_{>0}\int_0^{2\pi}|v(0,x)|^2\,dx\\
I_4 &= 2a\int_0^T(-\rho at+2\pi+\delta)|v(t,2\pi)|^2\,dt \geq -\underbrace{2a\left(\rho aT-2\pi-\delta\right)}_{>0}\int_0^T|v(t,2\pi)|^2\,dt\\
I_5 &= 2a\int_0^T(-\rho at+\delta)|v(t,0)|^2\,dt \geq -2a\int_0^T\underbrace{(\rho aT-\delta)}_{>0}|v(t,0)|^2
\end{aligned}$$

Using these estimates in (2.3.6), we obtain

$$\begin{aligned}
||e^{\alpha}D\varphi||_{L^2((0,T)\times(0,2\pi))} \geq K_1\int_0^T\int_0^{2\pi}|v(t,x)|^2\,dt\,dx + K_2\int_0^{2\pi}|v(T,x)|^2\,dx + K_3\int_0^{2\pi}|v(0,x)|^2\,dx\\
- K_4\int_0^T|v(t,2\pi)|^2\,dt - K_5\int_0^T|v(t,0)|^2\,dt
\end{aligned}$$

$$\begin{aligned}
&\int_0^T\int_0^{2\pi}|v(t,x)|^2\,dt\,dx + \int_0^{2\pi}|v(T,x)|^2\,dx + s\int_0^{2\pi}|v(0,x)|^2\,dx\\
&\qquad \leq \widetilde{C}\left(||e^{\alpha}D\varphi||_{L^2((0,T)\times(0,2\pi))} + \int_0^T\left(|v(t,0)|^2+|v(t,2\pi)|^2\right)dt\right)
\end{aligned}$$

Substituting $v(t,x) = e^{\alpha}\varphi(t,x)$ in the inequality and using the fact $\varphi(t,0) = \varphi(t,2\pi)$, we get the desired carleman estimate (2.3.5).

□

Proof of Theorem 2.3.1 Bounding the smooth weight function α in the above Carleman estimate (2.3.2) with φ as the solution of the adjoint system (2.1.3), we can easily get the desired observability inequalities as mentioned in (2.3.1), which proves the theorem.

□

Remark 2.3.3. *As argued in the Remark 2.2.3, we can conclude about the null controllability of transport equation as well.*

2.4 Negative controllability results

Theorem 2.4.1 (Boundary control system)**.** *Let $0 < T < \frac{2\pi}{|a|}$. Then, the transport equation* (1.0.1) *is not boundary null controllable at time T and hence not boundary exact controllable.*

Proof. We prove the theorem for $a = 1$, for simplicity, but similar argument can be used to get the result for any $a \in \mathbb{R} \setminus \{0\}\}$.

We know that the solution of transport equation is constant along its characteristics. Also from (2.3.1), we know that the null controllability of (1.0.1) is equivalent to the inequality

$$\int_0^{2\pi} |\varphi(0,x)|^2 \leq C \int_0^T |\varphi(t,0)|^2 \tag{2.4.1}$$

where φ is the solution of adjoint system.

Consider the following diagram, where the labels and shading in red represent the region where the solution of the adjoint system, φ is zero.

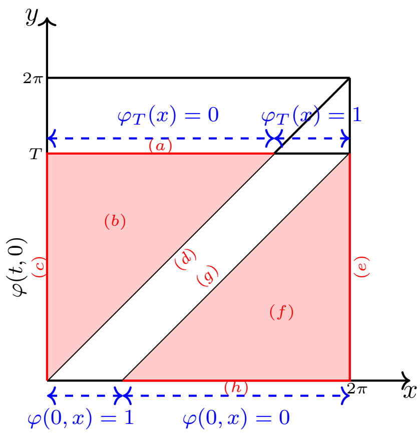

Figure 2.1: $a = 1$

Consider the terminal data φ_T as described in the picture. Using the characteristics, one can easily conclude that the solution $\varphi = 0$ in the shaded region and on the red lines $((a), (b), (c), (d), (e), (f), (g), (h))$. This forces $\varphi(0, x)$ to be 1 in one part and 0 in the remaining part as described in the picture above. Thus, the left hand side of (2.4.1) becomes nonzero, while the right hand side becomes zero, due to the line (c) and hence such choice of φ_T violates the observability inequality and hence the result follows.

□

Theorem 2.4.2 (Interior control system). *Let $0 < T < \frac{2\pi}{|a|}$. Then, the transport equation (1.0.1) is not null controllable with $\omega = (0, \xi)$ (if $a < 0$) or $\omega = (2\pi - \xi, 2\pi)$ (if $a > 0$) at time T and hence not boundary exact controlable, by means of localized interior.*

Proof. Assume $a > 0$. If possible, assume the statement of the theorem to be false. Now, extend the domain of space variable and consider the following system:

$$\begin{cases} \widetilde{\eta}_t + a\widetilde{\eta}_x = \mathbb{1}_\omega h, & (t,x) \in (0,T) \times (0,4\pi), \\ \widetilde{\eta}(t,0) = \widetilde{\eta}(t,4\pi), & t \in (0,T), \\ \widetilde{\eta}(0,x) = \widetilde{\eta}_0(x), & x \in (0,4\pi). \end{cases} \tag{2.4.2}$$

with $\omega = (4\pi - \xi, 4\pi)$. Then, the system is null controllable for some $T_0 = T + \frac{2\pi}{a} < \frac{4\pi}{a}$. Now, let $\eta = \widetilde{\eta}|_{(0,2\pi)}$. Then, η satisfies:

$$\begin{cases} \eta_t + a\eta_x = 0, & (t,x) \in (0,T) \times (0,2\pi), \\ \eta(t,0) = \eta(t,2\pi) + q(t), & t \in (0,T), \\ \eta(0,x) = \eta_0(x), & x \in (0,2\pi), \end{cases} \tag{2.4.3}$$

such that $\eta(T,x) = 0$, where $q(t) = \widetilde{\eta}(t,0) - \widetilde{\eta}(t,2\pi) \in L^2(0,T)$. This proves boundary null controllability of transport equation for $T < \frac{2\pi}{a}$, which is a contradiction to the Theorem 2.4.1. The fact that $\eta(T,x) = 0$ can be concluded using the characteristics as highlighted below in the picture for $a = 1$.

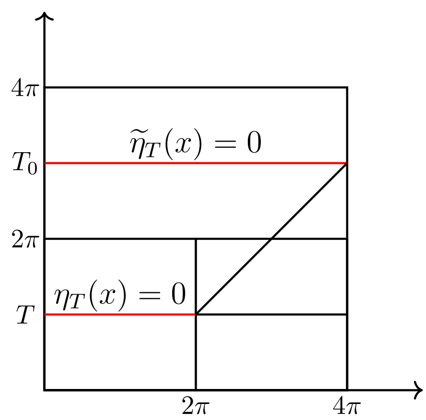

Figure 2.2: $a = 1$

Similar arguments can be used to prove the result for $a < 0$ with $\omega = (0, \xi)$.

□

CHAPTER 3

Null Controllability of KS-KdV Equation

3.1 Introduction

In this chapter we consider the linear parabolic equation (1.0.3) and study its controllability properties. We first establish the interior null controllability of the equation using a bilinear control function with localized domain in Section 3.2. In the next Section 3.3 we prove the boundary null controllability of the equation. For concluding the null controllability result in both the cases, we employ the method of moments, and find the explicit form of the control. As per the demand of the method, we first reduce the null controllability problem into an equivalent moment problem, and then using an available result regarding biorthogonal family from [30], we get the desired form of the control in both the cases.

Let us recall the system, posed in the space $(0,T)\times(0,2\pi)$:

$$\begin{cases} u_t+\gamma u_{xxxx}+u_{xxx}+\nu u_{xx}=0, & (t,x)\in(0,T)\times(0,2\pi), \\ u(t,0)=u(t,2\pi), u_x(t,0)=u_x(t,2\pi), & t\in(0,T), \\ u_{xx}(t,0)=u_{xx}(t,2\pi), u_{xxx}(t,0)=u_{xxx}(t,2\pi), & t\in(0,T), \\ u(0,x)=u_0(x), & x\in(0,2\pi), \end{cases} \tag{3.1.1}$$

where $\gamma,\nu>0$. The above system can be rewritten as

$$u'(t)=Au(t),\ \ t\in(0,T) \tag{3.1.2}$$

where, A is given by $Aw=-\gamma w_{xxxx}-w_{xxx}-\nu w_{xx}$ with $D(A)=H^4_p(0,2\pi)\subset L^2(0,2\pi)$.

For this system, we take the state space and control space respectively as

$$H=H^2_p(0,2\pi), U=L^2(0,T;L^2(0,2\pi))$$

We expand the initial data $u_0\in H^*$ in Fourier basis as: $u_0=\sum_{k\in\mathbb{Z}}a_k e^{ikx}$.

Now, let us consider the general control system to be considered in this thesis:

$$\begin{cases} u_t+\gamma u_{xxxx}+u_{xxx}+\nu u_{xx}=\mathbb{1}_\omega h, & (t,x)\in(0,T)\times(0,2\pi), \\ u(t,0)=u(t,2\pi)+q(t), u_x(t,0)=u_x(t,2\pi), & t\in(0,T), \\ u_{xx}(t,0)=u_{xx}(t,2\pi), u_{xxx}(t,0)=u_{xxx}(t,2\pi), & t\in(0,T), \\ u(0,x)=u_0(x), & x\in(0,2\pi), \end{cases} \tag{3.1.3}$$

The adjoint system of (3.1.3) can be given as:

$$-\varphi'(t) = A^*\varphi(t) \tag{3.1.4}$$

where, A^* is given by $A^*\varphi = -\gamma\varphi_{xxxx} + \varphi_{xxx} - \nu\varphi_{xx}$ with $D(A^*) = D(A) = H_p^4(0, 2\pi)$. The inhomogeneous adjoint system reads as

$$\begin{cases} \varphi_t - \gamma\varphi_{xxxx} + \varphi_{xxx} - \nu\varphi_{xx} = \tilde{h}, & (t,x) \in (0,T) \times (0,2\pi), \\ \varphi(t,0) = \varphi(t,2\pi),\ \varphi_x(t,0) = \varphi_x(t,2\pi), & t \in (0,T), \\ \varphi_{xx}(t,0) = \varphi_{xx}(t,2\pi),\ \varphi_{xxx}(t,0) = \varphi_{xxx}(t,2\pi), & t \in (0,T), \\ \varphi(T,x) = \varphi_T(x), & x \in (0,2\pi). \end{cases} \tag{3.1.5}$$

To give th well posedness result of (3.1.3) in H^*, we first give the well posedness result for the adjoint system in the space H. Let us denote $\mathbf{X} = L^2(0,T; L^2(0,2\pi))$ or $L^1(0,T;H)$.

Proposition 3.1.1 (Well posedness for (3.1.5))**.** *Let $\tilde{h} \in X, \varphi_T \in H$. Then, the adjoint system (3.1.5) has a unique solution $\varphi \in C([0,T]; H)$ which satisfies:*

$$||\varphi||_{C([0,T];H)\cap L^2(0,T;H^4)} \leq C\left(||\tilde{h}||_X + ||\varphi_T||_H\right). \tag{3.1.6}$$

Further, using trace regularity result, we get

$$||\varphi_{xxx}(\cdot,0)||_{L^2(0,T)} + ||\varphi_{xx}(\cdot,0)||_{L^2(0,T)} \leq C\left(||\tilde{h}||_X + ||\varphi_T||_H\right). \tag{3.1.7}$$

Proof can be found in the appendix (**??**).

Definition 3.1.2 (Solution of (3.1.3) in sense of transposition)**.** *Let $u_0 \in H^*, h \in L^2(0,T;H)$ and $q \in L^2(0,T)$. We say the function $u \in L^2(0,T;L^2(0,2\pi))$ is a solution of the system (3.1.3), if for any $\tilde{h} \in L^2(0,T;L^2(0,2\pi))$ and for every $t \in [0,T]$, the function $u(t,\cdot) \in L^2(0,2\pi)$ satisfies*

$$\int_0^T \int_0^{2\pi} u(t,x)\tilde{h}(t,x)\,dxdt = \langle u_0, \varphi(0,x)\rangle_{H^*,H} + \langle h, \varphi\rangle_{L^2(H^*),L^2(H)} \\ - \int_0^t \left(\gamma\,\overline{\varphi_{xxx}(s,0)} - \overline{\varphi_{xx}(s,0)} + \nu\,\overline{\varphi_x(s,0)}\right) q(s)\,ds,$$

where φ is solution of (3.1.5) with $\varphi_T = 0$.

Proposition 3.1.3 (Well-posedness of (3.1.3))**.** *Let $h \in L^2(0,T;H^*)$. Then for any $u_0 \in H^*$ and $q \in L^2(0,T)$, the system (3.1.3) has a unique solution $u \in C([0,T];H^*) \cap L^2(0,T;L^2(0,2\pi))$ in the sense of transposition as defined above.*

Proof. Using argument similar to Theorem 2.8 of [31], one can easily conclude this result, using Proposition 3.1.1.

□

Eigenvalues and Eigenvectors of $\mathbf{A^*}$: The eigen-equation corresponding to A^* is given as

$$A^*\varphi = \lambda\varphi,\ \ \lambda \in \mathbb{C},$$

equivalently,

$$\gamma\varphi_{xxxx} - \varphi_{xxx} + \nu\varphi_{xx} + \lambda\varphi = 0.$$

Expanding φ as Fourier series $\varphi = \sum_{k\in\mathbb{Z}} a_k e^{ikx}$, we get:

$$\gamma k^4 + ik^3 - \nu k^2 + \lambda = 0,$$

and so the eigenvalues of A^* are given by

$$\lambda_k = -\gamma k^4 + \nu k^2 - ik^3, \ k \in \mathbb{Z} \tag{3.1.8}$$

whose corresponding eigenvector is e^{ikx}, for $k \in \mathbb{Z}$.

Definition 3.1.4 (Biorthogonal family). *Let H be a Hilbert space and let $\{a_n\}, \{b_n\}$ be any two family of functions in H. We say $\{a_n\}$ is biorthogonal to $\{b_n\}$ if*

$$\langle a_n, b_k \rangle_H = \delta_{kn} = \begin{cases} 1, & \text{if } k = n, \\ 0, & else. \end{cases}$$

Lemma 3.1.5 (Biorthogonal family (see [30], Lemma 3.1)). *Let $\{\Lambda_m\}_{m\in\mathbb{N}}$ be a sequence of complex numbers such that, for some $\delta, \beta > 0$, it satisfies:*

$$\begin{cases} \mathsf{Re}(\Lambda_m) \geq \delta|\Lambda_m|, \forall m \in \mathbb{N}, \\ |\Lambda_m - \Lambda_n| \geq \beta\,|m-n|, \forall m,n \in \mathbb{N}, \\ \sum_{m=1}^{\infty} \frac{1}{|\Lambda_m|} < \infty. \end{cases} \tag{3.1.9}$$

Then, there exists a family of functions, $\{p_m\}_{k\in\mathbb{N}}$, biorthogonal to the family $\{e^{-\Lambda_m t}\}_{m\in\mathbb{N}}$ such that for every $\epsilon > 0$, there exist $C(\epsilon) > 0$ such that :

$$||p_m||_{L^2(0,\infty)} \leq C(\epsilon) e^{\epsilon \mathsf{Re}(\Lambda_m)} \tag{3.1.10}$$

Using the bijection between $\mathbb{N}$ and $\mathbb{Z}$, one can get the existence of biorthogonal family for a family, $\{e^{-\mu_k t}\}_{k\in\mathbb{Z}}$ using the above lemma, under the conditions mentioned in the corollary below:

Corollary 3.1.6. *Let $\{\mu_m\}_{m\in\mathbb{Z}}$ be a sequence of complex numbers such that, for some $\delta, \beta > 0$, it satisfies:*

$$\begin{cases} \mathsf{Re}(\mu_m) \geq \delta|\mu_m|, \forall m \in \mathbb{Z}, \\ |\mu_m - \mu_n| \geq \beta\,|m-n|, \forall m,n \in \mathbb{N}, \\ |\mu_m - \mu_n| \geq \beta\,|m-n|, \forall m,n \in \mathbb{Z}\setminus\mathbb{N}, \\ |\mu_m - \mu_n| \geq \beta\,|m+n|, \forall m \in \mathbb{N}, n \in \mathbb{Z}, \\ \sum_{m\in\mathbb{Z}} \frac{1}{|\mu_m|} < \infty. \end{cases} \tag{3.1.11}$$

Then, there exists a family of functions, $\{q_m\}_{k\in\mathbb{Z}}$, biorthogonal to the family $\{e^{-\mu_m t}\}_{m\in\mathbb{Z}}$ such that for every $\epsilon > 0$, there exist $C(\epsilon) > 0$ such that :

$$||q_m||_{L^2(0,\infty)} \leq C(\epsilon) e^{\epsilon\,\mathsf{Re}(\mu_m)} \tag{3.1.12}$$

Proof. Using the bijection between $\mathbb{N}$ and $\mathbb{Z}$, define:

$$\Lambda_m = \begin{cases} \mu_0, & \text{if } m = 1, \\ \mu_{m/2}, & \text{if } m \text{ is even}, \\ \mu_{(1-m)/2}, & \text{if } m \text{ is odd}. \end{cases}$$

Then, one can easily show that this family of Λ_m satisfies the hypothesis of the above lemma and so we get the existence of a family, $\{p_m\}_{m\in\mathbb{N}}$ biorthogonal to $\{e^{-\Lambda_m t}\}_{m\in\mathbb{N}}$ satisfying the estimate (3.1.10). Again using the bijection, we can define:

$$q_m = \begin{cases} p_1, \text{ if } m = 0, \\ p_{2m} \text{ if } m > 0, \\ p_{(-2m+1)} \text{ if } m < 0, \end{cases}$$

which satisfy $||q_m||_{L^2(0,\infty)} \leq C(\epsilon)e^{\epsilon \operatorname{Re}(\mu_m)}, \forall m \in \mathbb{Z}$.

□

3.2 Interior Controllability

Let us first write the concerned control system posed in $(0,T)\times(0,2\pi)$, for some $T > 0$:

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + \nu u_{xx} = \mathbb{1}_\omega h, & (t,x) \in (0,T)\times(0,2\pi), \\ u(t,0) = u(t,2\pi), u_x(t,0) = u_x(t,2\pi), & t \in (0,T), \\ u_{xx}(t,0) = u_{xx}(t,2\pi), u_{xxx}(t,0) = u_{xxx}(t,2\pi), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,2\pi), \end{cases} \tag{3.2.1}$$

where, $h(t,x) = f(x)g(t)$ is the bilinear interior control.

Theorem 3.2.1 (Interior null controllability of (3.1.1))**.** *Let $T > 0$, $\gamma > \nu$ and let ω be any open subset of $(0,2\pi)$. Then the system (3.1.1) is null controllable in H^* at time $T > 0$ by means of bilinear interior control of the form $h(t,x) = f(x)g(t)$ with its domain localized in ω.*

Reduction to moment problem : Assume the initial data u_0, terminal data φ_T, f, g to be smooth enough, so that the solution u, φ are smooth. Now, taking the duality product $\langle H^*, H\rangle$ in the equation with $\overline{\varphi}$ and then perform integration by parts to obtain

$$\langle u_t, \varphi\rangle_{H^*,H} = \langle u, A^*\varphi\rangle_{H^*,H} + \langle \mathbb{1}_\omega h, \varphi\rangle_{L^2}, \tag{3.2.2}$$

$$\text{i.e., } \frac{d}{dt}\left(\langle u, \varphi\rangle_{H^*,H}\right) = \int_\omega h(t,x)\overline{\varphi(t,x)}\,dx \tag{3.2.3}$$

On integrating the above equation w.r.t t over $[0,T]$, we get:

$$\langle u(T,\cdot), \varphi_T(\cdot)\rangle_{H^*,H} - \langle u_0(\cdot), \varphi(0,\cdot)\rangle_{H^*,H} = \int_0^T \int_\omega h(t,x)\overline{\varphi(t,x)}\,dxdt \tag{3.2.4}$$

The above equality holds even for $u_0 \in H^*, \varphi_T \in H, f \in L^2(0,2\pi)$, and $g \in L^2(0,T)$ by the density argument.

Claim 3.2.2. *The solution of the system (3.2.1) satisfy $(u(T,\cdot) = 0)$ (i.e., (3.1.1) is null controllable) iff for all $\varphi_T \in H$, the solution of adjoint system (3.1.5) satisfies*

$$\langle u_0(\cdot), \varphi(0,\cdot)\rangle_{H^*,H} = \int_0^T \int_\omega h(t,x)\overline{\varphi(t,x)}\,dxdt. \tag{3.2.5}$$

Proof. The 'only if' part is trivial from (3.2.4). For the 'if' part, assume (3.2.9) to be true, then by (3.2.4) we have

$$\langle u(T,\cdot), \varphi_T(\cdot)\rangle_{L^2} = 0, \ \forall \varphi_T \in H$$

and hence $u(T,\cdot) = 0$.

□

Take $\varphi_T(x) = e^{ikx}$ for $k \in \mathbb{Z}$, then the solution of the adjoint system (3.1.5) is given by

$$\varphi(t,x) = e^{\lambda_k (T-t)} e^{ikx}, \ (t,x) \in (0,T) \times (0,2\pi),$$

then substituting this solution in (3.2.9), we get:

$$\int_0^T \int_\omega f(x) g(t) e^{\overline{\lambda}_k (T-t)} e^{-ikx} \, dx dt = -\left\langle u_0(\cdot), e^{\lambda_k T} \varphi(0,\cdot)\right\rangle_{H^*,H},$$

which on simplification gives

$$f_k \int_0^T g(t) e^{-\overline{\lambda}_k t} \, dt = \left\langle u_0, e^{ikx}\right\rangle_{H^*,H}, \text{ for } k \in \mathbb{Z}. \tag{3.2.6}$$

Using the change of variable $t \mapsto (T-t)$, the above equality can be written as:

$$f_k \int_0^T g(t) e^{-(-\overline{\lambda}_k)t} \, dt = e^{\overline{\lambda}_k T} \left\langle u_0, e^{ikx}\right\rangle_{H^*,H}, \text{ for } k \in \mathbb{Z}. \tag{3.2.7}$$

Define $\mu_k = -\lambda_k = \gamma k^4 - \nu k^2 + ik^3$, then the above moment problem can be rewritten as

$$f_k \int_0^T g_1(t) e^{-(\overline{\mu}_k)t} \, dt = e^{-\overline{\mu}_k T} \left\langle u_0, e^{ikx}\right\rangle_{H^*,H}, \text{ for } k \in \mathbb{Z}, \tag{3.2.8}$$

where, $g_1(t) = g(T-t)$.

Note that as $\{e^{ikx}\}_{k\in\mathbb{Z}}$ forms a basis of H, so varying φ_T over this basis is enough, and thus we have following lemma:

Lemma 3.2.3 (Equivalent moment problem). *The solution of the system* (3.2.1) *satisfy* $u(T,\cdot) = 0$ *(i.e.,* (3.1.1) *is null controllable) iff there exist* $f \in L^2(0,2\pi)$ *with support in* ω*, and* $g_1 \in L^2(0,T)$ *satisfying*

$$f_k \left\langle g_1(t), e^{-\mu_k t}\right\rangle_{H^*,H} = e^{-\overline{\mu}_k T} \gamma_k, \ \textit{for } k \in \mathbb{Z}, \tag{3.2.9}$$

where, $\gamma_k = \left\langle u_0, e^{ikx}\right\rangle_{H^*,H} = \left\langle u_0, e^{ikx}\right\rangle_{L^2(0,2\pi)} = a_k$ *(coefficient in the series expansion) and* $f_k = \int_\omega f(x) e^{-ikx} \, dx$.

Remark 3.2.4. *Note that we need to ensure that* $f_k \neq 0$*, else we will have to put the corresponding condition on the initial data,* u_0*. More precisely, if* f_{k_0} *is zero, then we need to choose* $u_0 \in H$ *satisfying* $\left\langle u_0, e^{ik_0 x}\right\rangle_{H^*,H} = 0$.

Proof of Theorem 3.2.1. Note that proving this theorem is equivalent to solving the corresponding moment problem (3.2.9).

Step 1 : Construction of f. We construct $f \in L^2(\omega)$ such that $f_k \neq 0$. Let $\alpha \in \omega$ and $\rho \in (0,1)$ be a quadratic irrational (irrational number which is a root of quadratic equation with integral coefficients) such that $[\alpha, \alpha+\rho\pi]$ is subset of ω. Define

$$f(x) = \chi_{[\alpha,\alpha+\rho\pi]}(x), \quad \forall x \in (0,2\pi) \tag{3.2.10}$$

Clearly, $f \in L^2(0,2\pi)$ with support inside ω and also

$$f_0 = \rho\pi \neq 0$$
$$f_k = \int_0^{2\pi} f(x)e^{-ikx} = \frac{e^{-ik\alpha}}{ik}(1-e^{-ik\rho\pi}) \neq 0,\ \forall\, k \in \mathbb{Z}\setminus\{0\}$$

Now, as ρ is quadratic irrational so it can be approximated by rational numbers to order 2 and to no higher order ([32], Theorem 188), i.e., there exist $C > 0$ such that for any integers p and q, $q \neq 0$,

$$\left|\rho - \frac{p}{q}\right| \geq \frac{C}{q^2}. \tag{3.2.11}$$

Also, for $k \in \mathbb{Z}\setminus\{0\}$

$$\begin{aligned} |f_k| = \frac{1}{|k|}|1-e^{-ik\rho\pi}| &= \frac{1}{|k|}\left|2\,\sin^2\left(\frac{k\rho\pi}{2}\right) + i\,2\,\sin\left(\frac{k\rho\pi}{2}\right)\cos\left(\frac{k\rho\pi}{2}\right)\right| \\ &= 2\frac{\left|\sin\left(\frac{k\rho\pi}{2}\right)\right|}{|k|}. \end{aligned} \tag{3.2.12}$$

Note that,

$$\sin^2 x \geq \frac{4x^2}{\pi^2}, \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{3.2.13}$$

For any fixed $k \in \mathbb{Z}\setminus\{0\}$, choose $p \in \mathbb{Z}$ such that $0 \leq \frac{k\rho\pi}{2} - p\pi \leq \pi$.

Case I. If $0 \leq \frac{k\rho\pi}{2} - p\pi \leq \frac{\pi}{2}$, then we have

$$\begin{aligned} \sin^2\left(\frac{k\rho\pi}{2}\right) = \sin^2\left(\frac{k\rho\pi}{2} - p\pi\right) &\geq k^2\left(\rho - \frac{2p}{k}\right)^2, \text{ by (3.2.13)} \\ &\geq k^2\frac{C}{k^4} = \frac{C}{k^2} \text{ by (3.2.11)}. \end{aligned}$$

Case II. If $\frac{\pi}{2} \leq \frac{k\rho\pi}{2} - p\pi \leq \pi$, i.e., $-\frac{\pi}{2} \leq \frac{k\rho\pi}{2} - (p+1)\pi \leq 0$, then we have

$$\sin^2\left(\frac{k\rho\pi}{2}\right) = \sin^2\left(\frac{k\rho\pi}{2} - (p+1)\pi\right) \geq \frac{C}{k^2}, \text{ by last case.}$$

Combining the above two cases, we have

$$\sin^2\left(\frac{\pi}{2}k\rho\right) \geq \frac{C^2}{k^2},\ \forall\, k \in \mathbb{Z}\setminus\{0\}. \tag{3.2.14}$$

Plugging the estimate (3.2.14) in (3.2.12), we obtain $|f_k| \geq \frac{\widetilde{C}}{|k|^2}$ for some $\widetilde{C} > 0$, $k \in \mathbb{Z}\setminus\{0\}$.

Step 2 : Existence of biorthogonal family of $\mathbf{e}^{-\mu_k t}$**.** Recall $\mu_k = \gamma k^4 - \nu k^2 + ik^3$. Then, for $k \in \mathbb{Z}$, we have

$$|\mu_k| = |\gamma k^4 - \nu k^2 + ik^3| \leq |\gamma k^4 - \nu k^2| + |k^3| \leq |\gamma k^4 - \nu k^2| + C|\gamma k^4 - \nu k^2|, \text{ as } \gamma > \nu,$$

and so we get

$$\text{Re}(\mu_k) \geq \frac{1}{(C+1)}|\mu_k|$$

Also,

$$\sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{|\mu_k|} \leq \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{|k|^8} < \infty$$

Now, for $m, n \in \mathbb{Z}$

$$\begin{aligned} |\mu_m - \mu_n| &= |\gamma(m-n)(m^3 + nm^2 + mn^2 + n^3) - \nu(m-n)(m+n) + i(m-n)(m^2 + n^2 + mn)| \\ &\geq |m-n||\gamma(m^3 + nm^2 + mn^2 + n^3) - \nu(m+n) + i(m^2 + n^2 + mn)| \\ &\geq |m-n||m^2 + n^2 + mn| \geq 3|m-n|, \end{aligned}$$

for $m \in \mathbb{N}, n \in \mathbb{Z}$

$$\begin{aligned} |\mu_m - \mu_n| &= |\gamma(m+n)(m^3 - nm^2 + mn^2 - n^3) - \nu(m+n)(m-n) + i(m+n)(m^2 + n^2 - mn)| \\ &\geq |m+n||\gamma(m^3 - nm^2 + mn^2 - n^3) - \nu(m-n) + i(m^2 + n^2 - mn)| \\ &\geq |m-n||m^2 + n^2 - mn| \geq 3|m+n|, \end{aligned}$$

Thus from Corollary 3.1.6, we get the existence of a biorthogonal family, $\{q_m\}_{k\in\mathbb{Z}}$ of $\{e^{-\mu_m t}\}_{m\in\mathbb{Z}}$ such that for every $\epsilon > 0$, there exist $C(\epsilon) > 0$ satisfying:

$$||q_m||_{L^2(0,\infty)} \leq C(\epsilon) e^{\epsilon\,\text{Re}(\mu_m)}, \ \forall m \in \mathbb{Z} \tag{3.2.15}$$

Step 3 : Construction of g_1**.** Let us define g_1 formally as

$$g_1(t) = \sum_{k\in\mathbb{Z}} f_k^{-1} e^{-\mu_k T} \gamma_k\, q_k(t), \text{ for } t \in (0,T). \tag{3.2.16}$$

Then from Lemma 3.2.3, it is quite obvious that this formal definition of g_1 solves the moment problem (3.2.8). Now to make it precise, we just need to show that $g_1 \in L^2(0,T)$.

$$\begin{aligned} ||g_1||_{L^2(0,T)} &\leq \sum_{k\in\mathbb{Z}} \left|e^{-\mu_k T}\right| |f_k^{-1}|\, |\gamma_k|\, ||q_k||_{L^2(0,T)} \\ &\leq C(\epsilon) \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{-\text{Re}(\mu_k)T} \frac{|a_k|}{|k|^2} e^{\epsilon \text{Re}(\mu_k)} + |a_0|\rho\pi, \text{for any } \epsilon > 0 \\ &\leq C(\epsilon) \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|a_k|}{|k|^2} exp\left((T-\epsilon)\text{Re}(-\mu_k)\right), \text{for any } \epsilon > 0 \\ &< \infty, \text{ if } T > 0. \end{aligned}$$

Thus, the moment problem (3.2.9) can be solved for any $T > 0$, and hence the system (3.1.1) is null controllable in H^* at any time $T > 0$. The proof is complete.

□

Remark 3.2.5. *Note that in the Step 2, the summation term does not include μ_0 as it is zero. But in such cases, we can actually shift the eigenvalues by 1, using the transformation $\tilde{u} = e^{-t}u$ and then proving null controllability of $\tilde{u}$ will give the null controllability of u as exponential function can never vanish on real line.*

3.3 Boundary Controllability

Let $T > 0$. Now consider the following concerned boundary control system:

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + \nu u_{xx} = 0, & (t,x) \in (0,T) \times (0,2\pi), \\ u(t,0) = u(t,2\pi) + q(t), u_x(t,0) = u_x(t,2\pi), & t \in (0,T), \\ u_{xx}(t,0) = u_{xx}(t,2\pi), u_{xxx}(t,0) = u_{xxx}(t,2\pi), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,2\pi), \end{cases} \tag{3.3.1}$$

where, $q(t)$ is the boundary control.
We define the space $\dot{H}^* = \{w \in H^* : \langle w, 1 \rangle_{H^*,H} = 0\}$.

Theorem 3.3.1 (Boundary null controllability of (3.1.1))**.** *Let $T > 0$, and let $\gamma > \nu$. Then the system (3.1.1) is null controllable in $\dot{H}^*$ at time $T > 0$ by means of periodic boundary control acting through the zeroth derivative of u. That is, the solution u of above control system (3.3.1) satisfies $u(T) = 0$, for any given initial data $u_0 \in H^*$.*

Reduction to the moment problem: Let us assume that the initial data u_0 of (3.3.1), terminal data φ_T of (3.1.5), q are smooth enough, so that the solution u, φ are sufficiently smooth. Then, we take the duality product $\langle \cdot, \cdot \rangle_{H^*,H}$ in the equation with φ, and then perform integration by parts to get

$$\langle u(T,\cdot), \varphi_T(\cdot) \rangle_{H^*,H} - \langle u_0(\cdot), \varphi(0,\cdot) \rangle_{H^*,H} = -\int_0^T \left(\gamma\, \overline{\varphi_{xxx}(s,0)} - \overline{\varphi_{xx}(s,0)} + \nu\, \overline{\varphi_x(s,0)} \right) q(s)\, dxds$$

By the density argument, the above identity holds even if we take the initial data u_0, terminal data φ_T from the space H^*, H, respectively and the control q from $L^2(0,T)$. As argued in the last section (see Claim 3.2.2), one can easily conclude that the solution of system (3.3.1) satisfies $u(T) = 0$ iff the following holds

$$\langle u_0(\cdot), \varphi(0,\cdot) \rangle_{H^*,H} = \int_0^T \left(\gamma\, \overline{\varphi_{xxx}(s,0)} - \overline{\varphi_{xx}(s,0)} + \nu\, \overline{\varphi_x(s,0)} \right) q(s)\, ds \tag{3.3.2}$$

where φ is the solution of adjoint problem (3.1.5) with $\varphi_T \in H$. Again as done in the last section, we use the above equation and vary φ_T over the basis of H, consisting of eigenvectors of A^* to get the following moment problem.

Lemma 3.3.2 (Equivalent moment problem)**.** *Let $T > 0$. Then, the solution of (3.3.1) satisfies $u(T) = 0$ iff there exists $\tilde{q} \in L^2(0,T)$ which solves the following moment problem:*

$$\int_0^T \tilde{q}(t) e^{-\overline{\mu}_k t}\, dt = e^{-\overline{\mu}_k T} \frac{a_k}{k^2 + i(-\gamma k^3 + \nu k)} := e^{-\overline{\mu}_k T} \gamma_k, \;\; k \in \mathbb{Z} \setminus \{0\} \tag{3.3.3}$$

under the restriction $a_0 = 0$, where $a_k = \left\langle u_0, e^{ikx} \right\rangle_{H^*,H}$ and $\tilde{q}(t) = q(T-t)$.

Proof of Theorem 3.3.1 From the statement of the above lemma, it is sufficient to solve the moment problem (3.3.3) to prove the theorem. So, to solve the moment problem, let us define $\tilde{q}$ formally as

$$\tilde{q}(t) = \sum_{k \in \mathbb{Z}} e^{-\overline{\mu}_k T} \gamma_k q_k$$

where $\{q_k\}_{k\in\mathbb{Z}}$ is the biorthogonal family of $\{e^{-\mu_k t}\}_{k\in\mathbb{Z}}$, obtained in the last section while proving Theorem 3.2.1. This clearly solves the moment problem (3.3.3) formally. So, we only need to show $q \in L^2(0,T)$.

$$\begin{aligned}
||\tilde{q}||_{L^2(0,T)} &\leq \sum_{k\in\mathbb{Z}} |\gamma_k| e^{\mathsf{Re}(-\mu_k)T} ||q_k||_{L^2(0,T)} \\
&\leq C(\epsilon) \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|a_k|}{|k|^2} e^{(T-\epsilon)\mathsf{Re}(-\mu_k)} \\
&< \infty, \text{ if } T > 0.
\end{aligned}$$

□

CHAPTER 4

Conclusions and Future Works

In this thesis, we have seen the controllability of a linear hyperbolic equation and linear parabolic equation in Chapter 2 and Chapter 3, respectively. Let us now point out the difference between the control behavior of these considered control systems:

1. In the transport equation, we have proved the exact controllability of the system, which certainly implies null and approximate controllability. But in the case of KS-KdV equation, we have not even talked about exact controllability and just proved the result concerning the null controllability of the system. The KS-KdV equation actually has smoothing effect and so if we take any nonsmooth data, it will be smoothened after sometime and so the system cannot be exact controllable in the concerned space which is not C^∞. Thus, the question of exact controllability does not make any sense at all, and so we study null controllability of the system.

2. Also, note that the KS-KdV equation is null controllable at any time $T > 0$, but the transport equation can be controlled in time T iff $T > \frac{2\pi}{|a|}$.

My next plan is to study the controllability of the coupled system of transport and KS-KdV equation, i.e., a hyperbolic-parabolic coupled system. In this case, we do not have any result concerning the existence of biorthogonal family due to the hyperbolic branch of the eigenvalues (which is of the form $ik + d + O(k^{-1})$). So, if we want to follow the method of moments, we will have to construct the desired biorthogonal family. The same thing can also be proved using the Carleman estimates, but in that method the choice of proper weight function would be the challenge.

In the study of controllability of this coupled system, it is again obvious to not to question about the exact controllability due to the smoothing effect of the parabolic equation. So, it would be interesting to study the null controllability of the system and also to observe which equation's control property is being dominated in this case, more specifically, will the system have some restriction on the controllability time as in the case of transport equation or will it be controllable for any time $T > 0$?

Appendix A

Proof of Proposition 3.1.1

We assume both the system parameters, γ, ν to be 1 for simplicity in writing. Let us first perform a change of variable $t \mapsto T - t$ in (3.1.5) to get the forward adjoint system

$$\begin{cases} \varphi_t + \varphi_{xxxx} - \varphi_{xxx} + \varphi_{xx} = \tilde{h}, & (t,x) \in (0,T) \times (0,2\pi), \\ \varphi(0,x) = \varphi_T(x), & x \in (0,2\pi), \end{cases} \tag{A.0.1}$$

with the same periodic boundary condition as in (3.1.5).

Let us assume the data $\tilde{h}$ and φ_T to be regular enough. Then we multiply the equation (3.1.3) by $\overline{\varphi}$, and then take real part of the equation. Performing integration by parts and then adding them, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^{2\pi} |\varphi|^2 + \int_0^{2\pi} |\varphi_{xx}|^2 &= \text{Re}\left(\int_0^{2\pi} \tilde{h}\overline{\varphi}\right) - \text{Re}\left(\int_0^{2\pi} \varphi_{xx}\overline{\varphi}\right) \\ &\leq \int_0^{2\pi} |\tilde{h}\overline{\varphi}| + \int_0^{2\pi} |\varphi_{xx}\overline{\varphi}|. \end{aligned}$$

Using Young's inequality in the last term of R.H.S, we get

$$\frac{d}{dt}\int_0^{2\pi} |\varphi|^2 + \int_0^{2\pi} |\varphi_{xx}|^2 \leq 2\int_0^{2\pi} |\tilde{h}\overline{\varphi}| + \int_0^{2\pi} |\varphi|^2. \tag{A.0.2}$$

Next we multiply the first equation of the above forward adjoint system (A.0.1) by $\overline{\varphi}_{xxxx}$ and then consider the real part of the equation. Performing integration by parts, we obtain

$$\begin{aligned} &\text{Re}\left(\int_0^{2\pi} \varphi_{txx}\overline{\varphi}_{xx}\right) + \int_0^{2\pi} |\varphi_{xxxx}|^2 \\ &\qquad = -\text{Re}\left(\int_0^{2\pi} \varphi_{xx}\overline{\varphi}_{xxxx}\right) + \text{Re}\left(\int_0^{2\pi} \tilde{h}\overline{\varphi}_{xxxx}\right) \\ &\qquad \leq \int_0^{2\pi} |\varphi_{xx}\overline{\varphi}_{xxxx}| + \text{Re}\left(\int_0^{2\pi} \tilde{h}\overline{\varphi}_{xxxx}\right). \end{aligned}$$

Applying Young's inequality for the first term of R.H.S with $\epsilon > 0$, we have:

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}|\varphi_{xx}|^2+\int_0^{2\pi}|\varphi_{xxxx}|^2\le\frac{\epsilon}{2}\int_0^{2\pi}|\varphi_{xxxx}|^2+\frac{1}{2\epsilon}\int_0^{2\pi}|\varphi_{xx}|^2+\text{Re}\left(\int_0^{2\pi}\tilde{h}\overline{\varphi}_{xxxx}\right).$$

After simplifying, we deduce

$$\frac{d}{dt}\int_0^{2\pi}|\varphi_{xx}|^2+(2-\epsilon)\int_0^{2\pi}|\varphi_{xxxx}|^2\le C\int_0^{2\pi}|\varphi_{xx}|^2+2\,\text{Re}\left(\int_0^{2\pi}\tilde{h}\overline{\varphi}_{xxxx}\right). \tag{A.0.3}$$

Case 1: $\tilde{\mathbf{h}}\in\mathbf{L^2(0,T;L^2(0,2\pi))}$. Using Young's inequality in the right hand side of (A.0.2), we obtain

$$\frac{d}{dt}\int_0^{2\pi}|\varphi|^2+\int_0^{2\pi}|\varphi_{xx}|^2\le 2\int_0^{2\pi}|\varphi|^2+\int_0^{2\pi}|\tilde{h}|^2.$$

Multiplying the inequality by e^{-3t} and then integrating over $[0,s]\subset[0,T]$, we get:

$$\int_0^{2\pi}|\varphi(s,\cdot)|^2+\int_0^s\int_0^{2\pi}|\varphi|^2\le C\left(\int_0^T\int_0^{2\pi}|\tilde{h}|^2+\int_0^{2\pi}|\varphi_T|^2\right). \tag{A.0.4}$$

Using the inequality Re $(z)\le|z|$, for $z\in\mathbb{C}$ and Young's inequality in the last integral of (A.0.3), we get

$$\frac{d}{dt}\int_0^{2\pi}|\varphi_{xx}|^2+\int_0^{2\pi}|\varphi_{xxxx}|^2\le C\int_0^{2\pi}|\varphi_{xx}|^2+\int_0^{2\pi}|\tilde{h}|^2. \tag{A.0.5}$$

Multiplying (A.0.5) by e^{-Ct} and then integrating w.r.t t over $[0,s]\subset[0,T]$, for all $s\in[0,T]$ we get

$$\int_0^{2\pi}|\varphi_{xx}(s,x)|^2\,dx\le C\left(\int_0^T\int_0^{2\pi}(|\tilde{h}|^2)+\int_0^{2\pi}|(\varphi_T)_{xx}|^2\right). \tag{A.0.6}$$

Now integrating (A.0.5) on $[0,s]\subset[0,T]$, and using (A.0.6), we obtain

$$\int_0^s\int_0^{2\pi}|\varphi_{xxxx}|^2\le C\left(\int_0^T\int_0^{2\pi}|\tilde{h}|^2+\int_0^{2\pi}|(\varphi_T)_{xx}|^2\right). \tag{A.0.7}$$

Adding the inequalities (A.0.4), (A.0.6) and (A.0.7) to get

$$\int_0^{2\pi}|\varphi(s,\cdot)|^2+\int_0^{2\pi}|\varphi_{xx}(s,\cdot)|^2+\int_0^s\int_0^{2\pi}|\varphi|^2+\int_0^s\int_0^{2\pi}|\varphi_{xxxx}|^2\le C\left(\int_0^T\int_0^{2\pi}|\tilde{h}|^2+||\varphi_T||_H^2\right).$$

On taking supremum over $s\in[0,T]$ and using the equivalence of Sobolev norms, we get

$$||\varphi||_{L^2(0,T;H^4)\cap L^\infty(0,T;H)}\le C\left(||\tilde{h}||_{(L^2(0,T;L^2(0,2\pi)))}+||\varphi_T|||_H\right).$$

Case 2: $\tilde{\mathbf{h}}\in\mathbf{L^1(0,T;H)}$. From (A.0.2), we have:

$$\frac{d}{dt}\int_0^{2\pi}|\varphi|^2+\int_0^{2\pi}|\varphi|^2+\int_0^{2\pi}|\varphi_{xx}|^2\le 2\int_0^{2\pi}|\varphi|^2+2\int_0^{2\pi}|\tilde{h}\overline{\varphi}|.$$

Multiplying the inequality by e^{-2t} and then integrating w.r.t over $[0,s] \subset [0,T]$, we write

$$\int_0^{2\pi} |\varphi(s,\cdot)|^2 + \int_0^s \int_0^{2\pi} |\varphi|^2 \leq C\left(\int_0^T \int_0^{2\pi} |\tilde{h}\overline{\varphi}| + \int_0^{2\pi} |\varphi_T|^2\right)$$
$$\leq C\left(||\tilde{h}||_{L^1(L^2)}||\varphi||_{L^\infty(L^2)} + \int_0^{2\pi} |\varphi_T|^2\right). \tag{A.0.8}$$

Taking supremum in s over $[0,T]$, we obtain

$$||\varphi||^2_{L^\infty(0,T;L^2)} + ||\varphi||^2_{L^2(0,T;L^2)} \leq C\left(||\tilde{h}||_{L^1(0,T;L^2)}||\varphi||_{L^\infty(0,T;L^2)} + \int_0^{2\pi} |\varphi_T|^2\right). \tag{A.0.9}$$

Note that

$$\left(\int_0^{2\pi} |\varphi_T|^2\right)^{\frac{1}{2}} \leq ||\varphi||_{L^\infty(0,T;L^2)}.$$

So, using these in the last inequality (A.0.9), we get

$$\left(||\varphi||_{L^\infty(0,T;L^2)}\right)^2 \leq C\left(||\varphi||_{L^\infty(0,T;L^2)}\right)\left(||\tilde{h}||_{L^1(0,T;L^2)} + ||\varphi_T||_{L^2}\right),$$
$$\text{thus, } ||\varphi||_{L^\infty(0,T;L^2)} \leq C\left(||\tilde{h}||_{L^1(0,T;L^2)} + ||\varphi_T||_{L^2}\right). \tag{A.0.10}$$

From (A.0.9) , we have

$$\begin{aligned} ||\varphi||^2_{L^2(0,T;L^2)} &\leq C\left(||\tilde{h}||_{L^1(0,T;L^2)}||\varphi||_{L^\infty(0,T;L^2)} + \int_0^{2\pi} (|\varphi_T|^2)\right) \\ &\leq C\left(||\tilde{h}||_{L^1(0,T;L^2)}\,||\varphi||_{L^\infty(0,T;L^2)} + ||\varphi||^2_{L^\infty(0,T;L^2)}\right) \\ &\leq C\left(||\varphi||_{L^\infty(0,T;L^2)}\right)\left(||\varphi||_{L^\infty(0,T;L^2)} + ||\tilde{h}||_{L^1(0,T;L^2)}\right) \\ &\leq C\left(||\tilde{h}||_{L^1(0,T;L^2)} + ||\varphi_T||_{L^2}\right)^2 \text{ using (A.0.10)} \end{aligned}$$
$$\text{and so, } ||\varphi||_{L^2(0,T;L^2)} \leq C\left(||\tilde{h}||_{L^1(0,T;L^2)} + ||\varphi_T||_{L^2}\right) \tag{A.0.11}$$

Thus combining (A.0.10) and (A.0.11), we have

$$||\varphi||_{L^\infty(0,T;L^2)} + ||\varphi||_{L^2(0,T;L^2)} \leq C\left(||\tilde{h}||_{L^1(0,T;L^2)} + ||\varphi_T||_{L^2}\right) \tag{A.0.12}$$

Performing integration by parts in (A.0.3), we get:

$$\frac{d}{dt}\int_0^{2\pi} |\varphi_{xx}|^2 + \int_0^{2\pi} |\varphi_{xxxx}|^2 \leq C\int_0^{2\pi} |\varphi_{xx}|^2 + 2\,\mathrm{Re}\left(\int_0^{2\pi} (\tilde{h})_{xx}\varphi_{xx}\right)$$
$$\leq C\int_0^{2\pi} |\varphi_{xx}|^2 + 2\int_0^{2\pi} |(\tilde{h})_{xx}\varphi_{xx}|. \tag{A.0.13}$$

Again multiplying the inequality by e^{-Ct}, integrating w.r.t t over $[0,s] \subset [0,T]$ and then taking supremum over $s \in [0,T]$, we get

$$\sup_{s\in[0,T]} \int_0^{2\pi} (|\varphi_{xx}(s,\cdot)|^2) \leq C\int_0^T \int_0^{2\pi} |(\tilde{h})_{xx}\varphi_{xx}| + \int_0^{2\pi} |(\varphi_T)_{xx}|^2$$
$$\leq C\left(||(\tilde{h})_{xx}||_{L^1(0,T;L^2)}||(\varphi)_{xx}||_{L^\infty(0,T;L^2)}\right) + \int_0^{2\pi} |(\varphi_T)_{xx}|^2. \tag{A.0.14}$$

Using similar analysis as done above, we get:

$$||\varphi_{xx}||_{L^\infty(0,T:L^2)} \leq C||\tilde{h}_{xx}||_{L^1(0,T;L^2)} + ||(\varphi_T)_{xx}||_{L^2}. \tag{A.0.15}$$

Integrating (A.0.13) w.r.t t over $[0,T]$ and using inequality (A.0.15), we get:

$$||\varphi_{xxxx}||_{L^2(0,T;L^2)} \leq C\left(||\tilde{h}_{xx}||_{L^1(L^2)} + ||(\varphi_T)_{xx}||_{L^2}\right). \tag{A.0.16}$$

On adding (A.0.11), (A.0.15) and (A.0.16), we get:

$$||\varphi||_{L^2(0,T;H^4)\cap L^\infty(0,T;H)} \leq C\left(||\tilde{h}||_{L^1(0,T;H)} + ||\varphi_T|||_H\right).$$

Thus, combining both cases together we have

$$||\varphi||_{L^\infty([0,T],H)\cap L^2(0,T;H^4)} \leq C\left(||\tilde{h}||_{\mathbf{x}} + ||\varphi_T||_H\right),$$

for some $C > 0$.

The solution $\varphi \in L^2(0,T;H^4)$, so from the equation we get $\varphi_t \in L^2(0,T;L^2)$ and hence by the classical properties of these spaces we get $\varphi \in C([0,T];H)$ and hence the proof of (3.1.6) is complete. Using the continuous embedding of $H^4(0,2\pi)$ in $C^3([0,2\pi])$, we get (3.1.7).

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